Many-valued logic and sequence arguments in value theory

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Abstract

Some find it plausible that a sufficiently long duration of torture is worse than any duration of mild headaches. Similarly, it has been claimed that a million humans living great lives is better than any number of worm-like creatures feeling a few seconds of pleasure each. Some have related bad things to good things along the same lines. For example, one may hold that a future in which a sufficient number of beings experience a lifetime of torture is bad, regardless of what else that future contains, while minor bad things, such as slight unpleasantness, can always be counterbalanced by enough good things. Among the most common objections to such ideas are sequence arguments. But sequence arguments are usually formulated in classical logic. One might therefore wonder if they work if we instead adopt many-valued logic. I show that, in a common many-valued logical framework, the answer depends on which versions of transitivity are used as premises. We get valid sequence arguments if we grant any of several strong forms of transitivity of 'is at least as bad as' and a notion of completeness. Other, weaker forms of transitivity lead to invalid sequence arguments. The plausibility of the premises is largely set aside here, but I tentatively note that almost all of the forms of transitivity that lead to valid sequence arguments seem intuitively problematic. Still, a few moderately strong forms of transitivity that might be acceptable lead to valid sequence arguments, although weaker statements of the initial value claims avoid these arguments at least to some extent.

1 Introduction

Some find it plausible that there are values that cannot be counterbalanced by other values; for example, that a sufficiently large amount of torture is worse than any amount of mild headaches.[1] An example concerning positive value

[1] E.g., Carlson (2000, pp. 246–247). For discussion, see, e.g., Norcross (1997) and Schönherr (2018).

is provided by Lemos (1993, p. 487) who finds it better that a million people live excellent lives than that any number of worm-like creatures each feel a few seconds of pleasure.[2] One can relate bad things to good things along the same lines. For example, some authors seem sympathetic to the following idea: some horrible things such as a sufficiently large finite number of humans experiencing a lifetime of torment cannot be counterbalanced by various good things, regardless of the amount of those good things, while trivially bad things can always be counterbalanced by sufficiently many good things.[3]

These ideas are important for policy-making and the allocation of healthcare resources (Voorhoeve 2015). For example, should limited public funds be spent on treating many people with mild illnesses or a few with the worst health conditions? The ideas are also important for the impossibility theorems in population ethics (Carlson 2015; Thomas 2018).

I deal with some of the most common objections to such ideas, namely a group of similar objections called sequence arguments (or spectrum or continuum arguments), which have been much studied.[4] I will explain them in detail later, but the following is a sketch of a sequence argument against the view that a sufficiently large amount of torture is worse than any amount of mild headaches: There is a sequence of intermediate bads between torture and mild headache such as the following: torture, a terrible disease, a less serious disease, severe headache, moderate headache, mild headache. Spelt-out sequence arguments include more bads so that adjacent bads are more similar to each other. If a sufficiently large amount of torture is worse than any amount of mild headaches, there is a bad in the sequence such that this relation holds between it and its successor; for example, a sufficiently large amount of severe headaches is worse than any amount of moderate headaches. It is implausible, the argument goes, that this holds between adjacent bads in the sequence, which are so similar. Hence, the plausibility of the original view of torture versus mild headaches is undermined.

The main sequence arguments are formulated in classical logic, which assumes there are only two truth values, true and false, and that every declarative sentence is either true or false. I investigate whether sequence arguments are convincing if one instead uses many-valued logics; that is, logics with more than two truth values. More specifically, I focus on the validity of sequence arguments that use many-valued logic, and largely leave the plausibility of the premises for future research.

The truth values in many-valued logic are sometimes called truth degrees, and I assume, as is common, that they are numbers between 0 and 1, where 0 is falsest and 1 is truest. For example, in some many-valued logics, a sentence

[2] For more historical references, see Arrhenius (2005, p. 97).

[3] Such authors include Mayerfeld (1999, pp. 176–180), Brülde (2010, p. 577), Hedenius (1955, pp. 100–102), and Erik Carlson (e-mail to the author, Oct. 1, 2019).

[4] Early sequence arguments were formulated by Temkin (1996), Norcross (1997), and Rachels (1998). More recent work has been done by, e.g., Temkin (2012), Arrhenius and Rabinowicz (2015), Handfield and Rabinowicz (2018), Nebel (2018), Pummer (2018), and Jensen (2020).

can be true to degree 0.85.

It has been suggested that one can reply to sequence arguments by appealing to vagueness, and that one of the options is a theory of vagueness involving degrees of truth (Qizilbash 2005) or many-valued logic (Knapp 2007).[5] But the treatments of the topic have been brief, and in contrast to these works, I do not appeal to vagueness. I focus on the logic, and I leave it open whether vagueness has any role to play.

There are several reasons why it is worthwhile to investigate many-valued logic and sequence arguments.[6] Broadly speaking, many-valued logic seems at least as suitable for use in value theory as does two-valued (e.g., classical) logic, regardless of sequence arguments, but many-valued logic also has particular strengths when it comes to such arguments. More specifically, many-valued logic allows for gradual changes in the phenomenon at hand to be mirrored by gradual changes in degrees of truth.[7] For example, if someone who is going bald loses one more hair, it can become slightly truer that the person is bald. Similarly, slight changes in evaluatively relevant features can be mirrored by slight changes in the truth degree of value statements about that phenomenon. A related advantage of using many-valued logic in value theory is that it allows for a nuanced, precise repertoire of positions. For example, one can assign a truth value such as 0.76 to a view in value theory.

There are long-standing questions about how to understand or interpret degrees of truth, what they mean and what they are (e.g., Gottwald 2001, p. 4; Bradley 2009, p. 208; Smith 2008, §5.1). And there are many proposed answers (e.g., Smets and Magrez 1987; Paris 2000; Smith 2008, p. 211; Cintula, Fermüller, and Noguera 2017, §9). The answers do not affect the main results of this paper so I leave these questions open, and I do not defend or presuppose any one answer to these questions. Still, as background, I will now give a glimpse of how one might and might not understand degrees of truth. Authors such as Hájek (1998, pp. 2, 4) and Dubois and Prade (2001) distinguish truth degrees from probabilities (and I follow their lead here). If one assumes that possession of properties comes in degrees, one can identify degrees of truth with degrees of property possession. As Smith (2008, p. 211) puts it, "if Bob's degree of baldness is 0.3, then 'Bob is bald' is 0.3 true." We would deal with betterness or worseness rather than baldness, but the story could be similar: the holding of the relation of worseness between two items can come in degrees. Another option is to understand the truth degree an agent would give to a sentence as the ease with which the agent can accept the sentence (Paris 1997).

In §2, I explain the views to which sequence arguments are objections, and in §3, I describe previous sequence arguments. Then we turn to many-valued

[5] There is also a literature on many-valued logic and the sorites paradox (Paoli 2019), which has some resemblance to sequence arguments (Temkin 1996; Pummer 2018, §3; Asgeirsson 2019).

[6] See Paoli (2003, forthcoming) for defences of many-valued logic, and Smith (2008) for a defence of degrees of truth. For writings favourable to many-valued logic, see, e.g., Behounek (2006), Hájek (2007), and Novák and Perfilieva (2011). For objections to many-valued logic, see Paoli (2003, pp. 367–368) and Smith (2008) and the sources cited there.

[7] A similar point is made by Paoli (2003, pp. 364–365) in relation to the sorites paradox.

logic and sequence arguments. In §4, I present different approaches to sequence arguments using many-valued logic, and I motivate my strategy. I then describe my logical framework (§5). In §6 and §7, I consider premises in sequence arguments. Finally, §8 contains my formal results about sequence arguments, and §9 concludes.

2 The views targeted by sequence arguments

The ideas targeted by sequence arguments can and have been specified in different ways. My focus is on the view that there are bad things which are inferior to other bad things, where 'inferior to' is defined as follows:

> **Inferiority:** An object b is inferior to another object b' if and only if there is a number m such that m b-objects are worse than any number of b'-objects.[8]

There are different ways to specify what a bad b and m b-objects are, and what 'worse than' refers to. I will give a few examples, but the following specifications do not matter for my results: An object b could be an experience with a given unpleasantness that lasts for one second, and m b-objects could mean m such experiences. In general, I think of m b-objects as m objects of the same type as b. And 'we might think of objects of the same type as being identical in all value-relevant respects,' as Arrhenius and Rabinowicz (2015, p. 232) say. The term 'worse,' could refer to the value of outcomes or something being worse for an individual.

Although I focus on inferiority between bads, my points in this paper are equally relevant to the analogous superiority relation between goods,[9] and to the aforementioned views that relate bads to goods along the same lines.

3 Previous sequence arguments in more detail

In general terms, sequence arguments assume a finite sequence of goods $g_1, \ldots, g_n$ or bads $b_1, \ldots, b_n$, where n is a positive integer. The bad b_1 could, for example, be torture, and b_n could be some minor bad such as mild discomfort. Sequence arguments typically assume transitivity and sometimes completeness of a relation such as 'at least as good as.'[10] The classical notion of transitivity of 'at least as bad as,' which I denote $\preccurlyeq$, is that for all a, b and c, $a \preccurlyeq b$ and $b \preccurlyeq c$ together imply $a \preccurlyeq c$. And a standard, classical statement of completeness of $\preccurlyeq$ is that for all a and b, either $a \preccurlyeq b$ or $b \preccurlyeq a$.

[8] I draw on the formulation of weak superiority by Arrhenius and Rabinowicz (2015, p. 232).

[9] I define superiority as follows: A good g is superior to another good g' if and only if there is a number m such that m g-objects are better than any number of g'-objects (cf. Arrhenius and Rabinowicz 2015, p. 232).

[10] E.g., Norcross (1997), Arrhenius and Rabinowicz (2015), and Handfield and Rabinowicz (2018).

An example of a clear sequence argument that assumes classical logic is provided by Arrhenius and Rabinowicz (2015, p. 241).[11] It is perhaps the argument in the literature that is most similar to the sequence arguments I formulate, and it goes as follows: If 'is at least as bad as' is complete and transitive, and if b_1 is inferior to b_n, then the sequence contains a bad b_i that is inferior to the bad b_{i+1} that immediately follows it. If the sequence is chosen such that each item is only marginally better than the preceding item, it is implausible or counterintuitive that b_i would be inferior to the only marginally better b_{i+1}. Since this is a consequence of the assumption that b_1 is inferior to b_n, the plausibility of this assumption is undermined.

It is an open question whether it is a problem if there is inferiority or superiority between adjacent items in a sequence.[12] I set the question aside and assume that it is desirable to avoid inferiority and superiority between adjacent items.

I follow the same basic route of granting completeness and transitivity for the sake of argument, and I will see whether sequence arguments of this kind work if we assume many-valued logic. Hence, our premises will mainly be many-valued versions of completeness and transitivity.

There are other types of sequence arguments, but I set them aside. For example, arguments without transitivity can be found in Nebel (2018) and Pummer (2018, §3), and they are quite different from the arguments I focus on. Arrhenius and Rabinowicz (2015, p. 241) present a sequence argument without assuming completeness, which has a weaker conclusion than their argument above that uses completeness. Other examples are the sequence arguments by Handfield and Rabinowicz (2018), which allow indeterminacy or incommensurability.

4 Approaches to sequence arguments using many-valued logic

There are many choices to make when working with many-valued logic and sequence arguments. One choice is which logics to assume. There is a wide range of many-valued logics with different sets of truth values, notions of logical consequence, and connectives for 'and,' 'or,' 'implies,' etc. (e.g., Gottwald 2001). Another choice is which premises to use in the sequence arguments. There are, for instance, several different versions of completeness and transitivity in many-valued logic that could be used as premises.

In this section, I outline two broad approaches to these choices, and I motivate my strategy. Then, in §5, I describe the logics I choose to use (essentially,

[11] Their argument is about goods, but I rephrase it so that it is about bads because I focus on bads. Arrhenius has confirmed in conversation that they had classical logic in mind when they formulated their argument.

[12] See, e.g., Carlson (2000), Binmore and Voorhoeve (2003), Rabinowicz (2003), Arrhenius (2005, p. 108), Arrhenius and Rabinowicz (2005, p. 138, 2015, p. 238), Norcross (2009, pp. 85–88), and Klocksiem (2016).

the most common and simplest logics). Thereafter I turn to the versions of completeness and transitivity to be used as premises.

It is not clear which of the following two approaches is best, and hence I will use both approaches, one at a time. But I will emphasise the second approach more due to some of its advantages, which I will mention shortly.

The first approach is to start with one or more specific many-valued logics, with certain quantifiers and logical connectives. From the quantifiers and connectives in a logic, we can get versions of transitivity and completeness. For example, in the family L of Łukasiewicz logics I will work with, we can state transitivity of the many-valued relation $\preccurlyeq$ using the quantifier $\forall$ (for all), the conjunction $\wedge$ and the implication $\rightarrow$ as $\forall a \forall b \forall c((a \preccurlyeq b \wedge b \preccurlyeq c) \rightarrow a \preccurlyeq c)$. Then we can consider sequence arguments with that formula as a premise. An advantage of this approach is that we start with a systematically constructed logic, where quantifiers and connectives ideally correspond to the natural language expressions 'for all,' 'and,' 'or,' 'implies,' etc. in a reasonable way, and where connectives may be definable in terms of one another in a standard, intuitive way (see, e.g., Smith 2012). Regarding this first approach, I will use L in one technical result. Łukasiewicz logic is 'the most intensely researched many-valued logic,' according to Hähnle (2001, p. 323).

The second approach is to place conditions such as transitivity and completeness on many-valued relations such as $\preccurlyeq$, without first selecting specific many-valued logics such as those in L. For example, if we let $[\![\,]\!]$ denote the truth value of a statement, a reasonable transitivity condition might be that for all a, b and c, $\min([\![a \preccurlyeq b]\!], [\![b \preccurlyeq c]\!]) \leq [\![a \preccurlyeq c]\!]$. This is how versions of transitivity and completeness are often formulated in the literature on infinite-valued (fuzzy) preference relations (e.g., Dasgupta and Deb 2001). We can treat such transitivity and completeness conditions as meta-level restrictions, and we can reason in our metalanguage about, for example, what follows from them. An advantage of this approach is that we can easily work with a wider range of potentially interesting transitivity and completeness conditions, regardless of whether and how they could be stated as formulas using the connectives in specific logics such as those in L. A related advantage of this second approach is that it lends itself well to drawing general conclusions about many-valued logic and sequence arguments. A third advantage is that we bracket, at least at the present stage of inquiry, the big topic of which many-valued versions of connectives, such as conjunction, are suitable. Instead, we focus on value relations such as $\preccurlyeq$ and their formal properties (e.g., the transitivity conditions that may hold for $\preccurlyeq$). Since this paper is fundamentally about questions in value theory, the properties of value relations seem more crucial than the choice of logical connectives.

Along the lines of the second approach, I will state a few basic, common properties of a many-valued logic, and use the symbol 'M' to represent the family of logics with those properties. I then consider ten versions of transitivity and several notions of completeness. In the end, I formulate and prove technical results about sequence arguments for all logics in the family M.[13]

[13] I am grateful to a reviewer for suggesting essentially this approach.

When using the second approach, there are questions about how to formulate, select and assess the plausibility of the transitivity and completeness conditions that are to be used as premises in the sequence arguments. An idea in the literature is that one can make intuitive judgements about, for example, whether a transitivity condition is too restrictive (e.g., Dasgupta and Deb 1996, p. 307). But perhaps this requires a clearer statement of what it means that it is true to degree, say, $\frac{1}{3}$ that a is worse than b,[14] which is a question I leave open. So, to provide a more complete treatment that does not hinge on picking out plausible transitivity and completeness conditions based on an account of the degrees of truth of value statements, I allow, for the sake of argument, that someone who wants to formulate a sequence argument is free to use a range of transitivity and completeness conditions. And I present results about the validity of sequence arguments for this range of options.

5 Our logical framework

I use many-sorted many-valued first-order logics at the object level. At this level, we have, for example, many-valued predicates such as $\preccurlyeq$, connectives such as $\wedge$, and quantifiers such as $\forall$. I use sorted logics for convenience because we are dealing with three sorts of things: numbers, which I have represented by m, bads such as b, and quantities of bads such as m b-objects. At the meta level, I use classical logic and induction. For example, I use classical logic when I use proof by contradiction, and when I assume that it is either true to degree 1 that b is inferior to b' or it is not true to degree 1 that b is inferior to b'.

Our formal object-level language $\mathcal{L}$ is 3-sorted and contains the sorts $\sigma_{\mathbb{Z}^+}$, σ_B and σ_Q, which, intuitively, are about positive integers, bads, and quantities of bads, respectively. Each sort will be associated with a domain: $\sigma_{\mathbb{Z}^+}$, σ_B, and σ_Q will be associated with the domains $D_{\sigma_{\mathbb{Z}^+}}$, D_{σ_B}, and D_{σ_Q}, respectively (I will sometimes simply call the domains $\mathbb{Z}^+$, B, and Q). We can think of $D_{\sigma_{\mathbb{Z}^+}}$ as the set $\{1, 2, 3, \ldots\}$, D_{σ_B} as the set of bads $\{b_1, \ldots, b_n\}$, and D_{σ_Q} as containing the element 7 b_1-objects, the element 4 b_2-objects, and so on for all combinations of numbers in $D_{\sigma_{\mathbb{Z}^+}}$ and bads in D_{σ_B}. Each sort has a set of variables: $\mathcal{V}_{\mathbb{Z}^+} = \{k, m, n, k', m', n', \ldots\}$, $\mathcal{V}_B = \{b, b', b'', \ldots\}$ and $\mathcal{V}_Q = \{q, q', q'', \ldots\}$. Similarly, the sorts have the sets of individual constants $\mathcal{C}_{\mathbb{Z}^+}$, $\mathcal{C}_B$ and $\mathcal{C}_Q$, respectively. $\mathcal{L}$ includes the binary relation symbols $\prec$, $\preccurlyeq$ and $\sim$ of type $\langle \sigma_Q, \sigma_Q \rangle$. The intended readings of $\prec$, $\preccurlyeq$ and $\sim$ are 'is worse than,' 'is at least as bad as' and 'is equally bad as,' respectively. Because the relation symbols are of type $\langle \sigma_Q, \sigma_Q \rangle$, the relations named by them will be relations between elements of the domain D_{σ_Q}; for example (roughly speaking), 7 b_1-objects $\prec$ 4 b_2-objects. $\mathcal{L}$ also contains the binary function symbol f of type $\langle \sigma_Q, \sigma_{\mathbb{Z}^+}, \sigma_B \rangle$. The symbol f will be associated with a function that, due to the type of f, takes an element of $D_{\sigma_{\mathbb{Z}^+}}$ and an element of D_{σ_B} as inputs and outputs an element of D_{σ_Q}. We can think of the function named by f as simply taking a number and a bad as inputs and giving us a quantity of a bad such as 7 b_1-objects as output.

[14] Thanks to a reviewer for pressing this point.

The set of truth values will be either of the following: A finite set of equidistant rational numbers between 0 and 1, always including 0 and 1; that is,

$$\mathcal{W}_p \coloneqq \left\{ \frac{i}{p-1} : 0 \leq i \leq p-1 \right\}$$

for an integer $p \geq 2$, where $\coloneqq$ is definitional equality. For example, $\mathcal{W}_4 = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$. Or the infinite set of all real numbers between 0 and 1, including 0 and 1; that is,

$$\mathcal{W}_\infty \coloneqq [0,1]$$

(Gottwald 2017). '$\mathcal{W}$' represents any of $\mathcal{W}_p$ or $\mathcal{W}_\infty$.

I will use the perhaps most basic notion of models and logical consequence in many-valued logic. A conclusion is a logical consequence of the premises if and only if (iff) the conclusion is true to degree 1 whenever all premises are true to degree 1. We can find this notion of consequence in several important many-valued logics (Gottwald 2001, pp. 180, 249, 267, 291, 313, 386). As usual in first-order logic, the truth value of a sentence depends on the interpretation of the language which involves a structure that corresponds to the language (Conradie and Goranko 2015, ch. 4). More exactly, in many-sorted many-valued first-order logic, a structure $\mathcal{S}$ (containing domains, relations and functions) for a language $\mathcal{J}$ consists of the following:

- for each sort σ in $\mathcal{J}$, a domain D_σ in $\mathcal{S}$;
- for each constant symbol c in $\mathcal{J}$ of sort σ, an element $c^{\mathcal{S}}$ in D_σ;
- for each predicate symbol P in $\mathcal{J}$ of type $\langle \sigma_1, \ldots, \sigma_n \rangle$, a relation $P^{\mathcal{S}}$ on $D_{\sigma_1} \times \ldots \times D_{\sigma_n}$ (i.e., a mapping $P^{\mathcal{S}}$ associating a truth value with each tuple $\langle d_1, \ldots, d_n \rangle$ where $d_i \in D_{\sigma_i}$ for $i = 1, \ldots, n$);
- for each function symbol f in $\mathcal{J}$ of type $\langle \sigma_0, \ldots, \sigma_n \rangle$, a function $f^{\mathcal{S}} : D_{\sigma_1} \times \ldots \times D_{\sigma_n} \to D_{\sigma_0}$

(cf. Hájek 1998, §5.5; Manzano 1993; Gottwald 2001, pp. 22, 27; Lucas 2019).[15] The truth value of a sentence A in $\mathcal{S}$ is denoted $[\![A]\!]_{\mathcal{S}}$. We say that $\mathcal{S}$ is a model of A and write $\mathcal{S} \vDash A$ iff $[\![A]\!]_{\mathcal{S}} = 1$. For a set of sentences Σ, $\mathcal{S}$ is a model of Σ and we write $\mathcal{S} \vDash \Sigma$ iff $[\![B]\!]_{\mathcal{S}} = 1$ for each $B \in \Sigma$. We say that A is a logical consequence of Σ and write $\Sigma \vDash A$ iff $\mathcal{S} \vDash \Sigma$ implies $\mathcal{S} \vDash A$ for all $\mathcal{S}$. That is, $\Sigma \vDash A$ iff every model of Σ is a model of A. Finally, A is logically valid and we write $\vDash A$ iff $\mathcal{S} \vDash A$ for all $\mathcal{S}$ (see Gottwald 2001, §3, 249).

I am going to define the universal quantifier $\forall$ and the existential quantifier $\exists$ in the seemingly most common way in many-valued logic (e.g., Gottwald

[15]For the purpose of this paper, we want to avoid the complications that arise when there is an element in a domain that is not named by any constant symbol in the language. Similarly, we want to avoid that the symbols and the elements do not match in the sense that, for example, the constant symbol 'b_2' names the element b_4. To avoid these complications, I hereafter assume that each element in each domain is named by the corresponding constant symbol so that, for example, the symbol 'b_2' names the element b_2.

2001, pp. 26, 28, 250, 308; Urquhart 2001, p. 274; Malinowski 2007, pp. 49, 51; Bergmann 2008, ch. 14; Smith 2008, p. 65). In this way, $\forall$ and $\exists$ work as generalisations of the perhaps most common versions of conjunction and disjunction (respectively) in many-valued logic (e.g., Smith 2008, pp. 65, 67, 70).[16] I define $\forall$ and $\exists$ in this standard way with the minor modification that the variable and domain are of a sort. In the following definitions, x_σ is a variable of sort σ, and H is a well-formed formula with at most one free variable x_σ:

$$\llbracket \forall x_\sigma H \rrbracket_{\mathcal{S}} \coloneqq \inf \left\{ \llbracket H[x_\sigma/d] \rrbracket_{\mathcal{S}} : d^{\mathcal{S}} \in D_\sigma \right\};$$
$$\llbracket \exists x_\sigma H \rrbracket_{\mathcal{S}} \coloneqq \sup \left\{ \llbracket H[x_\sigma/d] \rrbracket_{\mathcal{S}} : d^{\mathcal{S}} \in D_\sigma \right\}.$$

$\left\{ \llbracket H[x_\sigma/d] \rrbracket_{\mathcal{S}} : d^{\mathcal{S}} \in D_\sigma \right\}$ is the set of truth values of H gotten when, for every $d^{\mathcal{S}}$ in the domain D_σ, each free occurrence of x_σ in H is replaced with the constant d that names $d^{\mathcal{S}}$. Given a set S, $\inf\{S\}$ is the infimum (greatest lower bound) of S. For example, let S be a subset of $\mathbb{R}$. If $\inf\{S\}$ exists, it is the largest $r \in \mathbb{R}$ such that for all $s \in S$, $r \leq s$. Similarly, $\sup\{S\}$ is the supremum (least upper bound) of S. I will not consider other definitions of the quantifiers in this paper because that would give us several different notions of inferiority (because inferiority contains universal and existential quantification) and more versions of transitivity and completeness (which contain universal quantification). We will already deal with many different logics and ten versions of transitivity, so we will have to leave an investigation of sequence arguments with different versions of the quantifiers for another time.

To save on notation, I will omit $^{\mathcal{S}}$ and $_{\mathcal{S}}$ when it is clear from the context what is meant and, for example, write $\llbracket\ \rrbracket$ instead of $\llbracket\ \rrbracket_{\mathcal{S}}$. And I will typically use the same notation for variables, constants, and objects in the domain; for example, k, m and n for variables of sort $\sigma_{\mathbb{Z}^+}$, constants in $\mathcal{C}_{\mathbb{Z}^+}$, and objects in the domain $\mathbb{Z}^+$.

I use the notation 'M' for the family of all logics with $\mathcal{W}$, $\vDash$, $\forall$ and $\exists$, as defined above. 'M_p' and 'M_∞' represent such families of logics with the sets of truth values $\mathcal{W}_p$ and $\mathcal{W}_\infty$, respectively.

'L' denotes the family of Łukasiewicz logics I deal with. L has any of the sets of truth values $\mathcal{W}$, and the notions of $\vDash$, $\forall$ and $\exists$ are as in M. So L falls within M. But L has specific propositional connectives, while it is unspecified which connectives the logics in M have.

Łukasiewicz logic is often presented as having available two disjunction connectives $\vee$ and $\veebar$, and two conjunction connectives $\wedge$ and &, (Hájek 1998, pp. 65, 67; Gottwald 2001, pp. 179–181, 2017; Metcalfe, Olivetti, and Gabbay 2009, p. 146; Marra 2013). The connectives of L are listed in Table 1. I omit some parentheses when writing formulas. As usual, negation has preference over disjunction and conjunction, which have preference over implication and biconditional. For example, I write $((\neg A) \wedge B) \to (C \vee D)$ as $\neg A \wedge B \to C \vee D$.

[16]More exactly, $\forall$ and $\exists$ are generalisations of the min-conjunction $\wedge$ and the max-disjunction $\vee$ (described in Table 1 below).

Connective	Definition	Truth function
$A \to B$		$[\![A \to B]\!] = \min(1, 1 - [\![A]\!] + [\![B]\!])$
$\neg A$		$[\![\neg A]\!] = 1 - [\![A]\!]$
$A \vee B$	$(A \to B) \to B$	$[\![A \vee B]\!] = \max([\![A]\!], [\![B]\!])$
$A \wedge B$	$\neg(\neg A \vee \neg B)$	$[\![A \wedge B]\!] = \min([\![A]\!], [\![B]\!])$
$A \veebar B$	$\neg A \to B$	$[\![A \veebar B]\!] = \min(1, [\![A]\!] + [\![B]\!])$
$A \mathbin{\&} B$	$\neg(A \to \neg B)$	$[\![A \mathbin{\&} B]\!] = \max(0, [\![A]\!] + [\![B]\!] - 1)$
$A \leftrightarrow B$	$(A \to B) \wedge (B \to A)$	$[\![A \leftrightarrow B]\!] = 1 - \lvert [\![A]\!] - [\![B]\!] \rvert$

Table 1: Propositional connectives of Łukasiewicz logic (L)

In the truth function for $\leftrightarrow$, $|\,|$ is absolute value.

Let me give a few remarks on how to understand some of the connectives in Table 1. I start by mentioning the similarity between the Łukasiewicz implication $\to$ and classical material implication, which we can denote $\to_{\mathsf{C}}$. Essentially, each of $A \to B$ and $A \to_{\mathsf{C}} B$ is true iff B is at least as true as A (see Smets and Magrez 1987). More precisely, $A \to B$ is completely true (true to degree 1) iff B is at least as true as A; and $A \to_{\mathsf{C}} B$ is true iff A is false while B is true, both A and B are false, or both A and B are true. When A is truer than B, which in the classical case means that A is true and B is false, $A \to_{\mathsf{C}} B$ is false. The situation is similar for $\to$ because when A is completely true and B is completely false (true to degree 0), $A \to B$ is completely false. More generally, when A is truer than B, $A \to B$ is less than completely true but also sensitive to how much truer A is than B in that $A \to B$ is less true the truer A is compared to B.

The connectives $\to$, $\neg$ and $\veebar$ are interdefinable as implication, negation and disjunction are in classical logic (Cignoli, D'Ottaviano, and Mundici 2000, pp. 78–79). And there is a standard duality between $\veebar$ and & as they are related via De Morgan laws such as $\vDash \neg(A \mathbin{\&} B) \leftrightarrow \neg A \veebar \neg B$, which we can read as saying that 'not both A and B' has the same truth value as 'either not A or not B' (Gottwald 2001, pp. 181, 184).

The disjunction $A \vee B$ is true (to degree 1) if and *only* if at least one of A and B is true (to degree 1), which is a property one might want at least one of the disjunction connectives to have. And there is a duality via De Morgan laws between $\vee$ and $\wedge$ (Gottwald 2001, p. 184).

There are other many-valued versions of the connectives, besides those in Table 1. For L and other many-valued logics, there are questions about which, if any, versions of the connectives are suitable for modelling natural language sentences containing 'if ..., then,' 'not,' 'or,' or 'and.' And there are lists of desired properties of the connectives.[17] I will not try to make progress on these issues in this paper. I will now merely briefly reply to a couple of objections about connectives in many-valued logic, including those in L, in order to motivate the use of many-valued logic and L.

[17] For more on these matters, see Gottwald (2001, pp. 5–6, 63–106, 391) and Smith (2008, pp. 67–70).

A common objection is that 'A and not A' should get truth value 0, but $[\![A \wedge \neg A]\!] = 0.5$ if $[\![A]\!] = 0.5$.[18] For example, let A represent the sentence 'Ann is bald,' and suppose that it is half-true. If we use $\wedge$ for 'and' and $\neg$ for 'not,' then 'Ann is bald and Ann is not bald' becomes half-true. But one might believe that such a contradiction should be completely false. Also, the disjunction $\veebar$ and the conjunction & might seem to behave strangely in some cases. For example, let A still represent 'Ann is bald,' and let B represent 'Bob is bald.' If $[\![A]\!] = [\![B]\!] = 0.5$, then $[\![A \veebar B]\!] = 1$, which may sound too high, and $[\![A \,\&\, B]\!] = 0$, which may seem too low. In other words, when it is half-true that Ann is bald and half-true that Bob is bald, it becomes completely true that Ann or Bob is bald, and completely false that Ann and Bob are bald, which might seem dubious.

I mention two replies to these objections. First, regarding $A \wedge \neg A$, there are other forms of the law of contradiction which one can accept even if one rejects that $[\![A \wedge \neg A]\!]$ is always 0 (Rescher 1969, pp. 143–148). Second, one can argue that sometimes $\wedge$ is a suitable formalisation of 'and' while in other cases & is appropriate; for example, that 'A and not A' should be formalised as $A \& \neg A$, which always has truth value 0 (Fermüller 2011, pp. 200–201). An analogous claim can be made about $\vee$ and $\veebar$ as alternative formalisations of 'or.'[19] For example, Paoli (forthcoming) argues that classical logic is ambiguous and collapses a distinction between two types of connectives. Classical disjunction, conjunction and implication can each be disambiguated in two kinds of ways; for example, classical disjunction can be disambiguated as $\vee$ or $\veebar$, and classical conjunction can be disambiguated as $\wedge$ or & (a formula may contain all of $\vee$, $\veebar$, $\wedge$ and &).

I use classical logic and induction at the meta level for two reasons: First, it is common to do so (Williamson 1994, p. 130; Gottwald 2001, pp. 6–7; Chakraborty and Dutta 2010, p. 1889; Dutta and Chakraborty 2016, p. 238). Second, the object and meta levels are about different matters. It seems reasonable that value statements such as 'a is worse than b' can have more than two truth values. But classical logic and induction may be suitable for whether a sentence has a given truth value or not, which kinds of proofs to accept, etc. In the metalanguage, I use '$\Rightarrow$' for implication in classical logic, and I have classical logic in mind when I write 'implies,' 'if ..., then,' 'iff,' 'for all,' 'there is,' etc. Even though I assume classical logic at the meta level, my sequence arguments are different from the classical sequence arguments in the literature. One difference is that the classical arguments assume that value statements such as 'a is better than b' does not have an intermediate truth value such as $\frac{1}{2}$, while I allow such truth values.

[18] See Fermüller (2011, pp. 199–200) and Smith (2017) which contains further references.

[19] Thanks to Erik Carlson for bringing up in an e-mail to the author that interpreting 'and' as $\wedge$ is more plausible in some situations while interpreting 'and' as & seems more appropriate when 'A and B' is a contradiction. Carlson made a similar point about $\veebar$.

6 Many-valued relations and completeness

In this section and the next, I deal with the premises in sequence arguments that use many-valued logic. I try to provide a range of options to someone who would like to present a sequence argument. Still, to focus my investigation on the sequence arguments that seem most interesting, I set a few options aside. So there are transitivity and completeness conditions in the literature that I will not attempt to use as premises in sequence arguments. In this section, I first say which value relations may be used in our sequence arguments, and then I quickly grant a few uncontroversial premises. I then turn to the use of completeness conditions as premises in sequence arguments. I list several such conditions from the literature, including the most common ones, and I assume that someone formulating a sequence argument may use all of these except one.

I grant that someone formulating a sequence argument is free to use all of the relations $\preccurlyeq$, $\prec$ and $\sim$. One might find $\prec$ and $\sim$ conceptually clearer than $\preccurlyeq$, and therefore avoid $\preccurlyeq$ or define $\preccurlyeq$ in terms of $\prec$ and $\sim$.[20] Or one might find it more parsimonious to take $\preccurlyeq$ as primitive and define $\prec$ and $\sim$ in terms of $\preccurlyeq$ (Hansson 2001, p. 322).

It is uncontroversial that any bad thing is equally bad as itself, at least as bad as itself, and not worse than itself. In other words, $\sim$ and $\preccurlyeq$ are reflexive and $\prec$ is irreflexive. For a many-valued binary relation R, these properties are commonly defined as follows:[21]

$$\text{Reflexivity} \coloneqq \text{for all } a, [\![aRa]\!] = 1;$$
$$\text{Irreflexivity} \coloneqq \text{for all } a, [\![aRa]\!] = 0.$$

A sequence argument may contain the premises that $\sim$ and $\preccurlyeq$ are reflexive and that $\prec$ is irreflexive, in the senses just defined, although these premises will only have a minor role in this paper.[22]

The most common definitions of completeness of the single relation $\preccurlyeq$ seem to be

$$\text{Completeness } (C_{\preccurlyeq}) \coloneqq \text{for all } a, b, [\![a \preccurlyeq b]\!] + [\![b \preccurlyeq a]\!] \geq 1;$$
$$\text{Strong completeness} \coloneqq \text{for all } a, b, \max([\![a \preccurlyeq b]\!], [\![b \preccurlyeq a]\!]) = 1$$

(Barrett and Pattanaik 1989, pp. 238–239; Llamazares 2005, p. 479; Fono and Andjiga 2007, p. 668). I will look at sequence arguments with $C_{\preccurlyeq}$ as a premise, but not strong completeness because it is too restrictive given that it rules out both $a \preccurlyeq b$ and $b \preccurlyeq a$ having intermediate truth values between 0 and 1. To get a feel for $C_{\preccurlyeq}$, note that $C_{\preccurlyeq}$ is equivalent to the following formula in L having truth value 1: $\forall a \forall b (a \preccurlyeq b \veebar b \preccurlyeq a)$. This formula reads 'for all a and b, $a \preccurlyeq b$ or $b \preccurlyeq a$,' which is simply a standard statement of completeness of $\preccurlyeq$.

[20]Thanks to a reviewer for bringing up this matter.

[21]E.g., see Ovchinnikov and Roubens (1991, p. 319) and Moretti, Öztürk, and Tsoukiàs (2016, p. 52). For other versions of reflexivity, see Dubois and Prade (1980, p. 73) and Dutta and Chakraborty (2015, p. 101).

[22]I thank Rupert McCallum and a reviewer for suggesting that I take $\sim$ to be reflexive.

Instead of dealing only with $\preccurlyeq$, one can formulate notions of completeness as connections between two or more of the relations $\preccurlyeq$, $\prec$ and $\sim$. I will now list a couple of such notions that I grant as premises in sequence arguments. The first such condition is

$$F := \text{for all } a, b, [\![a \prec b]\!] = 1 - [\![b \preccurlyeq a]\!]$$

(e.g., Banerjee 1994; Barrett and Pattanaik 1989, pp. 238–239; Llamazares 2005, p. 480). One can motivate F as follows: If negation has the truth function it has in L, which is seemingly the most common truth function for negation, one can read F as saying that $a \prec b$ is as true as not $b \preccurlyeq a$. Or one can think of F as saying that the truth value of $a \prec b$ and the truth value of $b \preccurlyeq a$ together exhaust the range of truth (they sum to 1, which represents maximal truth).

F is equivalent to the following formula in L having truth value 1:

$$F^{\mathsf{L}} := \forall a \forall b (a \prec b \leftrightarrow \neg b \preccurlyeq a).$$

For any relation R, $\neg aRb$ means $\neg(aRb)$.

One may want a notion of completeness for only $\prec$ and $\sim$, in which case the following might be used (Van de Walle, De Baets, and Kerre 1998, pp. 116–117):[23]

$$\text{Trichotomy} := \text{for all } a, b, [\![a \prec b]\!] + [\![b \prec a]\!] + [\![a \sim b]\!] = 1.$$

As with F, one can think of trichotomy as saying that the truth values of $a \prec b$, $b \prec a$, and $a \sim b$ together exhaust the range of truth values (since they sum to 1).

Whether reflexivity of $\sim$ and $\preccurlyeq$, irreflexivity of $\prec$, $C_{\preccurlyeq}$, F, F^{L} and trichotomy are ultimately plausible is beyond the scope of this paper. I assume for the sake of argument that someone who wants to formulate a sequence argument is free to use them as premises.

7 Transitivity of many-valued relations

There are many versions of transitivity of many-valued relations. Ten of them are listed in Table 2 (I have shortened some of the names).[24] There are more but these ten cover a fair bit of the ground, and I have tried to include those most relevant to sequence arguments. I consider these forms of transitivity mainly because they figure in the literature, to which I largely defer for conceptual discussion.[25] Because the focus of this paper is on the validity of sequence arguments, it is not necessary to consider the interpretation of or motivation for the versions of transitivity, yet I will nonetheless make some brief remarks about these matters.

[23] Thanks to a reviewer for suggesting the use of a trichotomy.

[24] T_1–T_8 are listed by Dasgupta and Deb (2001, p. 493); T_9 and T_{10} are from Tanino (1984, p. 119, 1990, p. 175) and Herrera-Viedma et al. (2004, p. 101).

[25] I thank a reviewer for this suggestion.

In this section, R is a many-valued binary relation, the formulations of transitivity are for all a, b and c in the domain, and $\_R\_$ is short for $[\![\_R\_]\!]$.

T_1	Probabilistic-sum transitivity	If $0 < aRb$ and $0 < bRc$, then $aRb + bRc - aRb \cdot bRc \leq aRc$
T_2	Max-transitivity	If $0 < aRb$ and $0 < bRc$, then $\max(aRb, bRc) \leq aRc$
T_3	Weighted mean transitivity	If $0 < aRb$ and $0 < bRc$, then there is $\lambda \in (0, 1)$ such that $\lambda \max(aRb, bRc)$ $+(1-\lambda) \min(aRb, bRc) \leq aRc$
T_4	Min-transitivity	$\min(aRb, bRc) \leq aRc$
T_5	Product transitivity	$aRb \cdot bRc \leq aRc$
T_6	Sensitive transitivity	If $0 < aRb$ and $0 < bRc$, then $0 < aRc$
T_7	Weak min-transitivity	If $bRa \leq aRb$ and $cRb \leq bRc$, then $\min(aRb, bRc) \leq aRc$
T_8	Δ-transitivity	$aRb + bRc - 1 \leq aRc$
T_9	Multiplicative transitivity	$aRb \cdot bRc \cdot cRa = aRc \cdot cRb \cdot bRa$
T_{10}	Additive transitivity	$aRb + bRc - \frac{1}{2} = aRc$

Table 2: Versions of transitivity from the literature on fuzzy preference relations

Observation 1. $T_1 \Rightarrow T_2 \Rightarrow T_3 \Rightarrow T_4 \Rightarrow T_5 \Rightarrow T_6$.

Dasgupta and Deb (2001, p. 493) mention this observation and refer to sources for proofs.

I will, in the next section, consider the validity of sequence arguments assuming any of T_1–T_8, or restricted forms of these versions of transitivity, regardless of whether these premises are plausible or not. Still, I will now provide some background and comment briefly on the possible rationale for and plausibility of some of the more important versions of transitivity. The purposes of this are to make the versions of transitivity more understandable, to explain why I set a couple of transitivity conditions (T_9 and T_{10}) aside, to explain why it is worthwhile to consider the restricted versions of transitivity, and to ultimately suggest directions for future research.

Min-transitivity (T_4) is perhaps the most widely used form of transitivity in many-valued logic. It is equivalent to the following formula in L having truth value 1: $\forall a \forall b \forall c(aRb \wedge bRc \rightarrow aRc)$. This equivalence holds even if the implication in the formula is not the Łukasiewicz implication in Table 1, as long as the implication has the degree ranking property: $[\![A \rightarrow B]\!] = 1$ iff $[\![A]\!] \leq [\![B]\!]$. It has been mentioned as a property that each implication operation should have, and the Łukasiewicz implication has it (Gottwald 2001, pp. 97, 181). The property can be seen as giving a rationale for why most of the versions of transitivity above are formulated in terms of $\leq$.

But T_4 has been criticised, for example, by Basu (1984, p. 215), who uses a counterexample, and suggests a version similar to T_3 as a fix. T_4 has also been criticised for being too restrictive, and the similar but weaker T_7 has been proposed instead (e.g., Barrett and Pattanaik 1989, pp. 239–240; Dasgupta and Deb 2001, p. 499).

T_8 is equivalent to the following formula in L having truth value 1: $\forall a \forall b \forall c(aRb \mathbin{\&} bRc \rightarrow aRc)$. That is, just like T_4 but with the conjunction & instead of $\wedge$. Similarly, we can state T_5 as a formula using the conjunction and implication in product logic (Gottwald 2001, pp. 292, 308).

The following is indicative commentary on the plausibility of the versions of transitivity. Eight of these forms of transitivity of $\preccurlyeq$ or $\prec$ seem problematic as premises in a sequence argument in our framework (T_1–T_6, T_9 and T_{10}). T_{10} would be unsuitable so I will not consider it more, because if $aRb + bRc > 1.5$, then $aRc > 1$, which is outside of our sets of truth values. T_1–T_6 and T_9 would seemingly be intuitively problematic premises because of the following case (cf. Barrett and Pattanaik 1985, p. 78): There are two bads b_1 and b_2. Hereafter, I write m b-objects as mb; for example, $5b_1$ is 5 b_1-objects. Let R represent $\preccurlyeq$ or $\prec$. Suppose $100b_1R100b_2$ and $100b_2R101b_1$ are at least $\frac{1}{4}$, which could be sensible if b_1 and b_2 are very different and neither appears clearly at least as bad as or worse than the other. Each of T_1–T_4 implies $100b_1R101b_1$ is at least $\frac{1}{4}$, T_5 implies it is at least $\frac{1}{16}$, and T_6 implies it is greater than 0. As long as $100b_1R100b_2 > 0$ and $100b_2R101b_1 > 0$, each of T_1–T_6 implies $100b_1R101b_1 > 0$. T_9 has this implication if we plausibly assume $101b_1R100b_1 > 0$ because the left-hand side of T_9 becomes greater than 0 so all numbers on the right-hand side must be greater than 0. These implications seem problematic. $100b_1R101b_1$ might plausibly be 0 (and more plausibly less than $\frac{1}{4}$ or $\frac{1}{16}$) because, since b_1 is something bad, fewer b_1-objects are not worse than or equally bad as more b_1-objects but less bad.

The counterexamples against versions of transitivity I have just put forth (except the technical point against T_{10}) involve comparisons between different amounts of the same type of bad (e.g., $100b_1R101b_1$). One can claim that even if all versions of transitivity in Table 2 are implausible, they are stronger than needed; that is, that sequence arguments only need weaker forms of transitivity as premises. More precisely, one can claim that sequence arguments only need transitivity for different types of bads such as b_1, b_2 and b_3, and I have not presented any counterexamples to such weaker forms of transitivity. One could weaken the forms of transitivity as in Table 3 so that they only hold for different types of bads (m, n and k are positive integers, and that b, b' and b'' are distinct means that $b \neq b'$, $b' \neq b''$ and $b \neq b''$):

T_5^r	Restricted product transitivity	If b, b' and b'' are distinct, then $mbRnb' \cdot nb'Rkb'' \leq mbRkb''$
T_6^r	Restricted sensitive transitivity	If b, b' and b'' are distinct, then if $0 < mbRnb'$ and $0 < nb'Rkb''$, then $0 < mbRkb''$

Table 3: Examples of restricted versions of transitivity

To save space, I do not list all ten restricted versions of transitivity, but all versions in Table 2 could be restricted in the analogous way. For any form of transitivity, I write ^{r} when it is restricted to distinct b, b' and b'' as in T_5^r and T_6^r.

The following case suggests that at least T_1^r–T_4^r seem intuitively problematic: Suppose $mb_1Rnb_2 = nb_2Rkb_3 = w \in (0, 0.5)$. T_1^r–T_4^r each implies $mb_1Rkb_3 \geq w$, but it might plausibly be lower because if mb_1Rnb_2 and nb_2Rkb_3 are equally close to false, it could perhaps be even closer to false that mb_1Rkb_3.

T_9 and T_9^r are equalities, but T_1–T_8 and T_1^r–T_8^r are not. Because T_9 and T_9^r are equalities, they postulate an exceptionally stringent relationship among the truth values of aRb, bRc, cRa, etc. I therefore set T_9 and T_9^r aside.

Overall, the seemingly most acceptable forms of transitivity we are left with are T_5^r, T_6^r, T_7, T_7^r, T_8 and T_8^r. The others seem more problematic, and a few seem so unsuitable that I hereafter set them aside (T_9, T_9^r, T_{10} and T_{10}^r).

8 Sequence arguments using many-valued logic

In this section, I consider sequence arguments assuming T_1–T_8 or T_1^r–T_8^r. I find that either of T_1–T_5 or T_1^r–T_5^r results in a valid sequence argument against the claim that it is true to degree 1 that the first object b_1 in the sequence is inferior to the last object b_n (Theorem 1). So does T_6 or T_6^r when the number of truth values is finite (Theorem 2), but not when it is infinite (Theorem 3). Hence, one can avoid sequence arguments if the number of truth values is infinite and merely T_6 or T_6^r is granted. Alternatively, someone sympathetic to inferiority can reply to these valid sequence arguments by saying that it need not be true to degree 1 that b_1 is inferior to b_n. It may be true to a high degree w less than 1. This reply does not help much if either of T_1–T_4 or T_1^r–T_4^r is granted because then there is a b_i in the sequence such that it is true to at least degree w that b_i is inferior to its successor b_{i+1} (Theorem 4). But one can avoid this upshot of sequence arguments if merely T_5, T_5^r, T_6 or T_6^r is granted because then it can be true to a high degree w that b_1 is inferior to b_n without it being the case for any object that it is true to at least degree w that it is inferior its successor (Theorem 5). T_7,T_7^r, T_8 and T_8^r generally do not result in a valid sequence argument, even if it is true to degree 1 that b_1 is inferior to b_n (Theorem 6), although T_7 and T_7^r may do so when there are only three truth values. I leave an investigation of the following kind of sequence arguments for future research

(I focus on stronger sequence arguments in this paper): if we grant one of the seemingly acceptable premises T_5^r, T_7 or T_7^r, and if it is true to a high degree w less than 1 that b_1 is inferior to b_n, must there be a b_i such that it is true to a counterintuitively high degree less than w that b_i is inferior to b_{i+1}?[26]

I assume the family of logics M in all of my theorems and the technical result in appendix H. I assume the family of Łukasiewicz logics L in one technical result (in appendix E). For the definitions of M and L, see §5. When I speak of reflexivity, irreflexivity, F, $C_{\preccurlyeq}$, trichotomy, T_1–T_8 or T_1^r–T_8^r, I assume they are meta-level conditions on the structures (as above, a structure is denoted $\mathcal{S}$). For example, if T_4 is assumed, we are considering only the class of structures in which T_4 holds; the structures that satisfy T_4.

Recall that $\mathcal{L}$ is our formal language with three sorts and symbols $\prec$, f, etc. as described in §5.

I use $\ll$ for the notion of 'is inferior to' I work with in this section. $\ll$ is an abbreviation defined as follows:

$$b \ll b' := \exists m \forall n (f(m, b) \prec f(n, b')).$$

Informally, I read $b \ll b'$ as 'there is a positive integer m such that m b-objects are worse than any number (in $\mathbb{Z}^+$) of b'-objects.'[27] I abbreviate $f(m, b)$ as mb, so we can write $b \ll b'$ as $\exists m \forall n (mb \prec nb')$. When I say 'is inferior to' without mentioning a truth degree, I mean that it is true to degree 1.

The first result is that, assuming M, F and that any of the transitivity conditions T_1–T_5 or T_1^r–T_5^r holds for $\preccurlyeq$, we get a valid sequence argument.

Theorem 1. *In M, if F holds and any of T_1–T_5 or T_1^r–T_5^r holds for the relation $\preccurlyeq$, then in any finite sequence of objects in which the first object is inferior to the last object, there is an object that is inferior to its successor.*

Proof in appendix A. In other words, Theorem 1 says that, assuming M, in every structure $\mathcal{S}$ for $\mathcal{L}$ in which F holds and any of T_1–T_5 or T_1^r–T_5^r holds for $\preccurlyeq$, and in which there is a finite sequence $b_1, \dots, b_n$ where $\mathcal{S} \vDash b_1 \ll b_n$, there is a b_i with $i \in \{1, \dots, n-1\}$ such that $\mathcal{S} \vDash b_i \ll b_{i+1}$. Theorem 1 is phrased as it is for readability, and the other theorems are phrased similarly for the same reason, but all could be stated in terms of $\mathcal{S}$, $\vDash$, $\ll$, etc. along the lines just indicated for Theorem 1.

Theorem 1 has the problem that at least T_1–T_5 and T_1^r–T_4^r seem problematic, or so I suggested in §7. But this is a matter of intuition and debatable. Regardless, T_5^r might be acceptable, so we have a valid sequence argument with potentially acceptable premises.

The forms of transitivity considered so far (T_1–T_5 and T_1^r–T_5^r) are fairly strong. The weaker T_6 and T_6^r result in a valid sequence argument when the number of truth values is finite, but not when it is infinite, as the next two theorems show.

[26] See the remark at the end of appendix F for more information.

[27] Thanks to Graham Leigh regarding the formulation of $\ll$.

Theorem 2. *In M_p, if F holds and T_6 or T_6^r holds for the relation $\preccurlyeq$, then in any finite sequence of objects in which the first object is inferior to the last object, there is an object that is inferior to its successor.*

Proof in appendix B. Theorems 3, 4 and 6 below deal only with unrestricted forms of transitivity because if the unrestricted form holds, so does the restricted form (i.e., for all $i \in \{1, 2, \dots, 10\}$, $T_i \Rightarrow T_i^r$).

Theorem 3. *In M_∞ there is a structure for $\mathcal{L}$ that satisfies F, $C_\preccurlyeq$, trichotomy, reflexivity of the relations $\preccurlyeq$ and $\sim$, irreflexivity of the relation $\prec$, and T_6 for $\preccurlyeq$, $\prec$ and $\sim$, and which contains a finite sequence of objects in which the first object is inferior to the last object, but in which no object is inferior to its successor.*

Proof in appendix C. Theorem 3 shows that, assuming M_∞, even if we grant quite a large number of conditions such as trichotomy and T_6 for all three value relations, we can still avoid the purportedly unappealing implications of inferiority. Note that in theorems 1, 2 and 4 we want to rely on few, weak premises, while in theorems 3, 5 and 6 we want to allow many, strong conditions.

Someone sympathetic to inferiority can reply to theorems 1 and 2 by saying that it need not be true to degree 1 that b_1 is inferior to b_n. It may be true to a high degree w less than 1. But the next theorem (Theorem 4) shows that, given F and any of T_1–T_4 or T_1^r–T_4^r for $\preccurlyeq$, if $[\![b_1 \ll b_n]\!] = w \in [0, 1]$, then there is a b_i in the sequence such that $[\![b_i \ll b_{i+1}]\!] \geq w$. So the upshot of the next theorem is that if one accepts the assumptions in it, one does not avoid sequence arguments by claiming that it is merely true to degree $w \in [0, 1)$ that the first object is inferior to the last.

Theorem 4. *In M, if F holds and any of T_1–T_4 or T_1^r–T_4^r holds for the relation $\preccurlyeq$, then for any $w \in [0, 1]$, and in any finite sequence of objects in which it is true to degree w that the first object is inferior to the last object, there is an object such that it is true to at least degree w that it is inferior to its successor.*

Proof in appendix D.

In appendix E, I explain how we could proceed and get a result similar to Theorem 4 if we were to use the first approach in §4 and start with a specific family of logics such as L.

The next theorem shows that if we grant merely T_5 or T_6, then, as long as there are at least 5 truth values, we can avoid sequence arguments in the following sense: it can be true to degree $w \in [\frac{3}{4}, 1)$ that the first object is inferior to the last object without there being any object such that it is true to at least degree w that it is inferior to its successor.

Theorem 5. *In M_∞ and $\mathsf{M}_{p \geq 5}$, there is a structure for $\mathcal{L}$ that satisfies F, $C_\preccurlyeq$, trichotomy, reflexivity of the relations $\preccurlyeq$ and $\sim$, irreflexivity of the relation $\prec$, and T_5 and T_6 for $\preccurlyeq$, $\prec$ and $\sim$, and which contains a finite sequence of objects in which it is true to degree $w \in [\frac{3}{4}, 1)$ that the first object is inferior to the last object, but in which there is no object such that it is true to at least degree w that it is inferior to its successor.*

Proof in appendix F. The theorem says '$[\frac{3}{4}, 1)$' because when the set of truth values is $\mathcal{W}_5$, $\frac{3}{4}$ is the greatest truth value less than 1. When the number of truth values is greater, we can let w be a greater number in $[\frac{3}{4}, 1)$.

The next and final theorem shows that T_7 and T_8 are generally not enough to get a sequence argument (so neither are T_7^r and T_8^r), even if it is true to degree 1 that the first object is inferior to the last. The theorem deals with T_7 and T_8 at the same time for brevity and because one might try to use several transitivity conditions as premises in one argument.

Theorem 6. *In M_∞ and $\mathsf{M}_{p\geq 4}$ there is a structure for $\mathcal{L}$ that satisfies F, $C_{\preccurlyeq}$, trichotomy, reflexivity of the relations $\preccurlyeq$ and $\sim$, irreflexivity of the relation $\prec$, and T_7 and T_8 for $\preccurlyeq$, $\prec$ and $\sim$, and which contains a finite sequence of objects in which the first object is inferior to the last object, but in which no object is inferior to its successor.*

Proof in appendix G. Theorem 6 is about when there are more than three truth values, which I find more interesting than the case of only three truth values, but one can tell from the proof that an almost identical structure satisfies T_8 in M_3. We can thereby get a result like Theorem 6 in M_3 about only T_8 instead of both T_7 and T_8. I leave it unanswered whether, assuming M_3, T_7 or T_7^r results in a valid sequence argument.

One may respond to theorems 3, 5 and 6, which show that one can avoid certain sequence arguments, by saying that $\preccurlyeq$, $\prec$ and $\sim$ have some counterintuitive properties in those simple structures. A reason why one might find them counterintuitive is that the truth values of the value statements are independent of the number of each type of bad in most cases. One may want to see a more reasonable way of making value comparisons that avoids sequence arguments. That is a fair point. The structures in theorems 3, 5 and 6 are very simple and merely meant to be sufficient for logical purposes. In appendix H, I present a more complex example structure with more reasonable value comparisons. In the end, one might very well want a different and perhaps even more complex way of making value comparisons. My aims with this example structure are merely to point out a direction towards making reasonable value comparisons which avoid at least some type of sequence argument and to illustrate how one can confirm that such a way of making value comparisons does not violate some reasonable conditions (I use reflexivity of $\sim$ and $\preccurlyeq$, irreflexivity of $\prec$, F, $C_{\preccurlyeq}$ and T_8 for $\preccurlyeq$ as examples of such conditions).

In this example structure, it is true to degree 0.7 that the first bad is inferior to the last, but there is no bad such that it is true to at least degree 0.7 that it is inferior to its successor. I assume M_∞, and my structure contains the three bads b_1, b_2 and b_3. $[\![b_1 \ll b_3]\!] = 0.7$, but $[\![b_1 \ll b_2]\!] = [\![b_2 \ll b_3]\!] = 0.5$. The truth degrees of value comparisons depend on the quantities of the bads, which is one respect in which this structure is more intuitive than those in the proofs of theorems 3 and 6. Value comparisons in terms of $\prec$ have the following truth values: If $m \geq n$, $[\![mb_1 \prec nb_3]\!] = 1$; that is, a given number of b_1-objects are definitely worse than fewer or the same number of b_3-objects. If $m < n$, then for any fixed m, $[\![mb_1 \prec nb_3]\!]$ decreases and approaches a limit, which we can call w, as n increases. This resembles existing ideas of diminishing marginal value (e.g., Carlson 2000; Binmore and Voorhoeve 2003; Rabinowicz 2003), but, importantly, the intuition is not that additional b_3-objects contribute less and less disvalue to the whole. Rather, the intuition is that for a given number m of b_1-objects, it is true to some degree w that m b_1-objects are worse than any number of b_3-objects. And while it should become less true that m b_1-objects are worse than n b_3-objects as n increases, it should always be true to at least degree w. For a higher fixed m, the limit, which we can call w', is higher (i.e., $w < w'$). The intuition is that for a higher m, it is truer that m is a sufficient number of b_1-objects for this collection of b_1-objects to be worse than any number of b_3-objects. As m and then n approach infinity, $[\![mb_1 \prec nb_3]\!]$ approaches 0.7; that is, $[\![b_1 \ll b_3]\!] = 0.7$. Value comparisons of b_1-objects to b_2-objects and of b_2-objects to b_3-objects work analogously, except that the truth value 0.5 instead of 0.7 is approached.

9 Concluding remarks

My findings are partly good news and partly bad news for inferiority and similar views such as those in §1 and §2. My findings are bad news in that we get valid sequence arguments if we grant a form of completeness and any of several strong forms of transitivity (T_1–T_4 or T_1^r–T_4^r). The weaker forms of transitivity T_5 and T_5^r result in valid sequence arguments when it is true to degree 1 that the first object b_1 in the sequence is inferior to the last object b_n, and so do the even weaker T_6 and T_6^r when it is true to degree 1 that b_1 is inferior to b_n and the number of truth values is finite.

However, my findings are good news in that one can readily formulate arguments suggesting that all of the just mentioned forms of transitivity, except T_5^r and T_6^r, are intuitively problematic. And even if T_5, T_5^r, T_6 and T_6^r are granted as premises, one can, at least to some extent, avoid the purportedly unappealing implications of inferiority by holding that it is merely true to some high degree less than 1 that b_1 is inferior to b_n. Or if merely T_6 and T_6^r are granted as premises, one can avoid sequence arguments by holding that there are infinitely many truth values. The seemingly acceptable forms of transitivity T_7, T_7^r, T_8 and T_8^r are generally not enough to get a valid sequence argument. If there are only three truth values, T_7 and T_7^r may result in a valid sequence argument, but I would prefer to use more than three truth values. The path to a convincing sequence argument in our logical framework looks narrow.

We get the most convincing sequence arguments when we use the moderately strong forms of transitivity as premises. In particular, the most promising path to a convincing sequence argument seems to be to use T_5^r as a premise; perhaps T_7 or T_7^r could also be used. To make a sequence argument in our framework convincing, a reasonable step would be to argue extensively for the plausibility of using T_5^r (or perhaps T_7 or T_7^r) as a premise.[28] Another reasonable step is

[28]E.g., an objection to the use of T_5^r or T_7^r could be that it is ad hoc to restrict transitivity

to investigate, more thoroughly than I have done, what constraints T_5^r, T_7 and T_7^r put on the truth values of inferiority relationships in sequences, including in long sequences, which could result in the following forms of sequence arguments, which are weaker than the ones I have considered: If it is true to degree, say, 0.95 that b_1 is inferior to b_n, even if there need not be any b_i in the sequence such that is true to at least degree 0.95 that b_i is inferior to b_{i+1}, perhaps there must be a b_i such that the truth value of that b_i is inferior to b_{i+1} must be counterintuitively high.[29] Such forms of sequence arguments are yet to be explored.[30]

A Proof of Theorem 1

We can establish Theorem 1 using the following lemma and induction (cf. Arrhenius and Rabinowicz 2015, p. 241):

Lemma 1. *In M, if F holds and any of T_1–T_5 or T_1^r–T_5^r holds for the relation $\preccurlyeq$, then for any distinct objects b, b' and b'', if b is inferior to b'', then b is inferior to b' or b' is inferior to b''.*

Proof. Suppose b, b' and b'' are distinct. Let $w_1 := [\![b \ll b']\!]$ and $w_2 := [\![b' \ll b'']\!]$. Suppose $[\![b \ll b'']\!] = 1$ but $w_1, w_2 \in [0, 1)$. Pick $\varepsilon \in (0, 1)$ such that $w_1 + \varepsilon < 1$ and $w_2 + \varepsilon < 1$. Let $y := w_1 + \varepsilon$ and $z := w_2 + \varepsilon$. Pick m such that $[\![\forall k(mb \prec kb'')]\!] > y + z - y \cdot z$. There is such an m because $y + z - y \cdot z < 1$ and, by the assumption $[\![b \ll b'']\!] = 1$ and the definitions of $\ll$ and $\exists$, $\sup \{[\![\forall k(mb \prec kb'')]\!] : m \in \mathbb{Z}^+\} = 1$. To see that $y + z - y \cdot z < 1$, note that $1 - (y + z - y \cdot z) = (1 - y)(1 - z) > 0$, so $y + z - y \cdot z$ must be less than 1. By the definition of $\forall$, for all k,

(1) $[\![mb \prec kb'']\!] > y + z - y \cdot z.$

Pick n such that

(2) $[\![mb \prec nb']\!] < y.$

There is such an n because $[\![b \ll b']\!] = w_1 < y$ and, by the definitions of $\ll$, $\exists$ and $\forall$, for all m there is an n such that $[\![mb \prec nb']\!] < y$. Analogously, pick k such that

conditions so that they only hold for different types of bads. Thanks to Magnus Vinding for mentioning this.

[29] See the first paragraph of §8 and the remark at the end of appendix F.

[30] I am grateful for comments on earlier versions by Roger Crisp, Kaj Börge Hansen, Francesco Paoli, Nils Sylvan and Alex Voorhoeve. I thank the ALOPHIS group at the University of Cagliari, the LSE Choice Group, the PhD seminar in practical philosophy at Stockholm University, and Theron Pummer for helpful discussion. My supervisors Gustaf Arrhenius and Krister Bykvist have kindly contributed in many ways. The following people have been exceptionally helpful: Erik Carlson, Valentin Goranko, Laurenz Hudetz, Graham Leigh, Rupert McCallum, Karl Nygren, Daniel Ramöller and Magnus Vinding. Two anonymous reviewers gave very useful comments, and one of them was unusually generous and gave many detailed, skilled comments. I am grateful for thoughts on an ancestor to this paper from Campbell Brown, Jens Johansson, Anna Mahtani and Wlodek Rabinowicz. Thanks to Gunnar Björnsson and Mozaffar Qizilbash for answering questions related to my research.

(3) $[\![nb' \prec kb'']\!] < z$.

By (1), (2), (3) and F,

$$w_3 := [\![kb'' \preccurlyeq nb']\!] = 1 - [\![nb' \prec kb'']\!] > 1 - z;$$
$$w_4 := [\![nb' \preccurlyeq mb]\!] = 1 - [\![mb \prec nb']\!] > 1 - y;$$
$$w_5 := [\![kb'' \preccurlyeq mb]\!] = 1 - [\![mb \prec kb'']\!] < 1 - (y + z - y \cdot z).$$

$w_3 \cdot w_4 > (1-z)(1-y) = 1 - (y + z - y \cdot z) > w_5$, which contradicts T_5 and T_5^r of $\preccurlyeq$, which imply $w_3 \cdot w_4 \leq w_5$. By Observation 1, T_1–T_4 and T_1^r–T_4^r are contradicted too. Assuming classical logic at the meta level, we have a proof by contradiction of Lemma 1. □

We use Lemma 1 in the following induction on the length of the sequence to establish Theorem 1: Base step: The sequence contains two objects. If the first object is inferior to the last object, the first object is inferior to its successor. Induction hypothesis: When the length of the sequence is n objects ($n \geq 2$), if the first object is inferior to the last object, there is an object in the sequence that is inferior to its successor. Induction step: The length is $n+1$ objects. Suppose the first object is inferior to the last object (object $n+1$). If object n is inferior to object $n+1$, an object is inferior to its successor. If object n is not inferior to object $n+1$, then, by Lemma 1, the first object is inferior to object n. By the induction hypothesis, there is an object in the sequence that is inferior to its successor.

B Proof of Theorem 2

We can establish the theorem by a lemma and induction. The induction is the same as in appendix A except that Lemma 2 is used so I omit the induction.

Lemma 2. *In M_p, if F holds and T_6 or T_6^r holds for the relation $\preccurlyeq$, then for any distinct objects b, b' and b'', if b is inferior to b'', then b is inferior to b' or b' is inferior to b''.*

Proof. Suppose b, b' and b'' are distinct and $[\![b \ll b'']\!] = 1$. $[\![b \ll b'']\!] = 1$ iff $\sup\{[\![\forall k(mb \prec kb'')]\!] : m \in \mathbb{Z}^+\} = 1$ so because there are finitely many truth values, there is an m such that $[\![\forall k(mb \prec kb'')]\!] = 1$ and, by the definition of $\forall$, such that $[\![mb \prec kb'']\!] = 1$ for all k. By F,

(1) there is an m such that $[\![kb'' \preccurlyeq mb]\!] = 0$ for all k.

Case 1. $[\![b' \ll b'']\!] < 1$. Thus, for all n, there is a k such that $[\![nb' \prec kb'']\!] < 1$ and, by F, such that $[\![kb'' \preccurlyeq nb']\!] > 0$. So, by (1), there is an m such that for any choice of n, there is a k such that $[\![kb'' \preccurlyeq nb']\!] > 0$ and $[\![kb'' \preccurlyeq mb]\!] = 0$. By T_6 or T_6^r of $\preccurlyeq$, $[\![nb' \preccurlyeq mb]\!] = 0$ and, by F, $[\![mb \prec nb']\!] = 1$. So there is an m such that for any n, $[\![mb \prec nb']\!] = 1$; that is, $[\![b \ll b']\!] = 1$.

Case 2. $[\![b \ll b']\!] < 1$. Hence, for all m, there is an n such that $[\![mb \prec nb']\!] < 1$ and, by F, such that $[\![nb' \preccurlyeq mb]\!] > 0$. So, by (1), there is an m and an n such that $[\![nb' \preccurlyeq mb]\!] > 0$ and $[\![kb'' \preccurlyeq mb]\!] = 0$ for all k. By T_6 or T_6^r of $\preccurlyeq$, $[\![kb'' \preccurlyeq nb']\!] = 0$ for all k; hence, by F, $[\![nb' \prec kb'']\!] = 1$ for all k. So $[\![b' \ll b'']\!] = 1$. □

C Proof of Theorem 3

Let $\mathcal{S}$ contain the domains $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, $B = \{b_1, b_2, b_3\}$ and $Q = \mathbb{Z}^+ \times B$, and the function $f : \mathbb{Z}^+ \times B \to Q$, which is simply a bijection that maps each ordered pair $\langle m, b\rangle$ in $\mathbb{Z}^+ \times B$ to the same ordered pair $\langle m, b\rangle$ in Q. For all $m, n \in \mathbb{Z}^+$ and $b \in B$, let

$$[\![mb_1 \prec nb_3]\!] = w;$$
$$[\![mb_1 \sim nb_3]\!] = [\![nb_3 \sim mb_1]\!] = [\![nb_3 \preccurlyeq mb_1]\!] = 1 - w;$$
$$[\![mb_1 \preccurlyeq nb_3]\!] = [\![mb_1 \preccurlyeq nb_2]\!] = [\![mb_2 \preccurlyeq nb_3]\!] = 1;$$
$$[\![nb_3 \prec mb_1]\!] = [\![nb_2 \prec mb_1]\!] = [\![nb_3 \prec mb_2]\!] = 0;$$
$$[\![mb_1 \prec nb_2]\!] = [\![mb_2 \prec nb_3]\!] = w';$$
$$[\![nb_2 \preccurlyeq mb_1]\!] = [\![nb_3 \preccurlyeq mb_2]\!] = 1 - w';$$
$$[\![mb_1 \sim nb_2]\!] = [\![nb_2 \sim mb_1]\!] = [\![mb_2 \sim nb_3]\!] = [\![nb_3 \sim mb_2]\!] = 1 - w';$$
$$[\![mb \prec nb]\!] = 0;$$
$$[\![mb \preccurlyeq nb]\!] = [\![mb \sim nb]\!] = 1;$$

where $w = 1 - \frac{1}{2m}$ and $w' = \frac{1}{2}$. For example, $[\![mb_1 \sim nb_3]\!] = \frac{1}{2m}$.

That was the description of $\mathcal{S}$. In $\mathcal{S}$, b_1 is inferior to b_3 (i.e., $\mathcal{S} \vDash b_1 \ll b_3$) because $\sup\{[\![\forall n(mb_1 \prec nb_3)]\!] : m \in \mathbb{Z}^+\} = 1$. It is easy to confirm the following: there are no other inferiority relationships, $\prec$ is irreflexive, $\preccurlyeq$ and $\sim$ are reflexive, and F, $C_{\preccurlyeq}$, trichotomy, and T_6 for $\preccurlyeq$, $\prec$ and $\sim$ hold in $\mathcal{S}$. Confirming T_6 is the most complicated task so let us do that here. To violate T_6 for a relation R, we need the consequent $[\![aRc]\!]$ of T_6 to not be greater than 0. In $\mathcal{S}$, $\preccurlyeq$ and $\sim$ always map to truth values greater than 0, so T_6 holds for $\preccurlyeq$ and $\sim$. To violate T_6 for $\prec$, both parts of the antecedent of T_6 need to be greater than 0. We only get that with $[\![mb_1 \prec nb_2]\!]$ and $[\![nb_2 \prec kb_3]\!]$, where $m, n, k \in \mathbb{Z}^+$, in the antecedent, in which case we get $[\![mb_1 \prec kb_3]\!]$ in the consequent, which is greater than 0, so T_6 holds for $\prec$.

D Proof of Theorem 4

The proof of Theorem 4 is similar to the proof of Theorem 1 in appendix A. We start with the following lemma:

Lemma 3. *In M, if F holds and any of T_1–T_4 or T_1^r–T_4^r holds for the relation $\preccurlyeq$, then for any $w \in [0, 1]$ and any distinct objects b, b' and b'', if it is true to degree w that b is inferior to b'', then it is either true to at least degree w that b is inferior to b' or true to at least degree w that b' is inferior to b''.*

Proof. The proof of Lemma 3 is very similar to the proof of Lemma 1 in appendix A, so I mainly note the differences. Suppose $[\![b \ll b'']\!] = w \in (0, 1]$ and $w_1, w_2 \in [0, w)$. Pick $\varepsilon \in (0, 1)$ such that $w_1 + \varepsilon < w$ and $w_2 + \varepsilon < w$. Let $y := w_1 + \varepsilon$ and $z := w_2 + \varepsilon$. Pick m such that $[\![\forall k(mb \prec kb'')]\!] > \max(y, z)$.[31] (2) and (3) are the same as in appendix A, but (1) is different:

(1) $[\![mb \prec kb'']\!] > \max(y, z)$.

As in appendix A, we get the following, where the only difference from appendix A is that here we have $w_5 < 1 - \max(y, z)$:

$$\begin{aligned} w_3 &:= [\![kb'' \preccurlyeq nb']\!] = 1 - [\![nb' \prec kb'']\!] > 1 - z; \\ w_4 &:= [\![nb' \preccurlyeq mb]\!] = 1 - [\![mb \prec nb']\!] > 1 - y; \\ w_5 &:= [\![kb'' \preccurlyeq mb]\!] = 1 - [\![mb \prec kb'']\!] < 1 - \max(y, z). \end{aligned}$$

Note that $\min(w_3, w_4) > \min(1 - z, 1 - y) = 1 - \max(y, z) > w_5$, which contradicts T_4 and T_4^r for $\preccurlyeq$, which imply $\min(w_3, w_4) \leq w_5$. By Observation 1, T_1–T_3 and T_1^r–T_3^r are also contradicted. □

We can then establish Theorem 4 by the following induction on the length of the sequence, which is similar to the induction in appendix A:[32] Base step: The sequence contains two objects. If it is true to degree w that the first object is inferior to the second, it is true to degree w that the first is inferior to its successor. Induction hypothesis: When the length of the sequence is n objects, if it is true to degree w that the first object is inferior to the last object, there is an object in the sequence such that it is true to at least degree w that it is inferior to its successor. Induction step: The length is $n + 1$ objects. Suppose it is true to degree w that the first object is inferior to the last object (object $n + 1$). If it is true to at least degree w that object n is inferior to object $n + 1$, then there is an object such that it is true to at least degree w that it is inferior to its successor. If it is not true to at least degree w that object n is inferior to object $n + 1$, then, by Lemma 3, it is true to at least degree w that the first object is inferior to object n. By the induction hypothesis, there is an object in the sequence such that it is true to at least degree w that it is inferior to its successor.

E Using the first approach and starting from Łukasiewicz logic (**L**)

The purpose of this appendix is to illustrate a use of the first approach in §4 by starting from L and its connectives. We do not need the content of this appendix

[31] I am grateful to a reviewer for pointing out how one can prove Theorem 4 similarly to how Theorem 1 was proved by, among other things, using $\max(y, z)$ instead of $y + z - y \cdot z$.

[32] Thanks to Valentin Goranko for suggesting that one can do induction on the length of the sequence.

for the main results of this paper because we already have Theorem 4, which is a more general result than what we get in this appendix.

Suppose that instead of starting our investigation of sequence arguments with premises such as F and the versions of transitivity in tables 2 and 3, we start with notions of completeness and transitivity formulated using the connectives of L (see Table 1), for example, the following (from §6 and §7):

$$F^{\mathsf{L}} := \forall q \forall q' (q \prec q' \leftrightarrow \neg q' \preccurlyeq q);$$
$$T_4^{\mathsf{L}} := \forall q \forall q' \forall q'' (q \preccurlyeq q' \wedge q' \preccurlyeq q'' \rightarrow q \preccurlyeq q'').$$

We wonder what ramifications such premises have for inferiority among bads in a sequence. To keep with the spirit of building from the connectives of L, we could formulate

$$I := \forall b \forall b' \forall b'' (b \ll b'' \rightarrow b \ll b' \vee b' \ll b''),$$

and then a lemma in L:

Lemma 4. *$F^{\mathsf{L}}, T_4^{\mathsf{L}} \vDash I$.*

The following is an outline of a proof of Lemma 4: Suppose (1) $[\![F^{\mathsf{L}}]\!] = 1$; (2) $[\![T_4^{\mathsf{L}}]\!] = 1$; (3) $[\![I]\!] < 1$. By the definition of $\forall$ and the semantics of the connectives, (1) is equivalent to F, (2) is equivalent to T_4, and (3) iff there are b, b' and b'' such that $[\![b \ll b'']\!] > \max([\![b \ll b']\!], [\![b' \ll b'']\!])$. Let $w := [\![b \ll b'']\!]$, $w_1 := [\![b \ll b']\!]$ and $w_2 := [\![b' \ll b'']\!]$, and then reason as in the proof of Lemma 3 to get a contradiction. Assuming classical logic at the meta level, we have a proof by contradiction of that $[\![F^{\mathsf{L}}]\!] = 1$ and $[\![T_4^{\mathsf{L}}]\!] = 1$ imply $[\![I]\!] = 1$.

Because Lemma 4 is very similar to Lemma 3, we could use induction as in appendix D to get a result similar to Theorem 4, but with L instead of M and with T_4^{L} instead of T_1–T_4 and T_1^r–T_4^r. That is, we could conclude: In L, if F^{L} and T_4^{L} hold (true to degree 1), then for any $w \in [0, 1]$, and in any finite sequence of objects in which it is true to degree w that the first object is inferior to the last object, there is an object such that it is true to at least degree w that it is inferior to its successor.

F Proof of Theorem 5

We need to show that for each of the infinite number of logics in the families M_∞ and $\mathsf{M}_{p\geq 5}$, there is at least one structure with the properties listed in Theorem 5. We do that by letting each structure be the same as in appendix C, except that here our definitions of w and w' are different from the definitions of w and w' in appendix C. When the number p of truth values is finite and at least five (i.e., $\mathcal{W}_{p\geq 5}$), let w be the greatest truth value less than 1, and let w' be the greatest truth value less than w. For example, when the set of truth values is

$\mathcal{W}_5 = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$, $w = \frac{3}{4}$ and $w' = \frac{2}{4}$. In other words, for $\mathcal{W}_{p\geq 5}$, let

$$w = \frac{p-2}{p-1};$$
$$w' = \frac{p-3}{p-1}.$$

When the set of truth values is $\mathcal{W}_\infty$, we can, for simplicity, let $w = \frac{9}{10}$ and $w' = \frac{8}{10}$ just as for $\mathcal{W}_{11}$.

For $\mathcal{W}_{p\geq 5}$, we have $w \in [\frac{3}{4}, 1)$ and $w' < w$, which is all we need to use in most of the proof (except when we confirm T_5 and T_6 for $\preccurlyeq$ and $\sim$). As in appendix C, the only non-trivial part of the proof is to confirm transitivity, so I omit the other parts of the proof.

By confirming T_5, we also confirm T_6 because, by Observation 1, $T_5 \Rightarrow T_6$. T_5 holds for $\prec$ for essentially the same reason as T_6 holds for $\prec$ in appendix C: to violate T_5, both of the factors on the left-hand side of T_5 would need to be greater than 0, but then we would get $w' \cdot w' \leq w$, which holds because $w' < w$ and $w, w' \in [0, 1]$.

It remains to confirm T_5 for $\preccurlyeq$ and $\sim$. I use the notation that R represents $\preccurlyeq$ and $\sim$, $m, n, k \in \mathbb{Z}^+$ and $b, b' \in B$.

Case 1. $[\![\_bR\_b]\!]$ is the form of at least one of the factors in T_5 or the right-hand side of T_5.

Subcase 1a. $[\![mbRnb]\!] \cdot [\![nbRkb']\!] \leq [\![mbRkb']\!]$.

Subcase 1b. $[\![mb'Rnb]\!] \cdot [\![nbRkb]\!] \leq [\![mb'Rkb]\!]$.

Subcase 1c. $[\![mbRnb']\!] \cdot [\![nb'Rkb]\!] \leq [\![mbRkb]\!]$.

T_5 holds in subcases 1a and 1b because for all $w_1, w_2 \in [0, 1]$, $w_1 \cdot w_2 \leq w_2$ and $w_1 \cdot w_2 \leq w_1$. T_5 holds in subcase 1c because $[\![mbRkb]\!] = 1$.

Case 2. $[\![\_bR\_b]\!]$ is not the form of any of the factors in T_5 or the right-hand side of T_5. To violate T_5, the right-hand side of T_5 must be less than 1.

Subcase 2a. The right-hand side of T_5 is $1-w'$. To violate T_5, the left-hand side of T_5 would need to be greater than $1 - w'$, so both of the factors on the left-hand side of T_5 would need to be greater than $1 - w'$; that is, both would need to be 1. But, except for $[\![\_bR\_b]\!]$, only $[\![mb_1 \preccurlyeq nb_3]\!]$, $[\![mb_1 \preccurlyeq nb_2]\!]$ and $[\![mb_2 \preccurlyeq nb_3]\!]$ equal 1, and the only combination of them that could be on the left-hand side of T_5 is $[\![mb_1 \preccurlyeq nb_2]\!] \cdot [\![nb_2 \preccurlyeq kb_3]\!]$. But then we get $[\![mb_1 \preccurlyeq kb_3]\!] = 1$ on the right-hand side of T_5, so T_5 holds.

Subcase 2b. The right-hand side of T_5 is $1 - w$.

The rest of the proof is about subcase 2b. There are three ways in which the right-hand side of T_5 can be $1 - w$:

$$[\![mb_1 \sim nb_2]\!] \cdot [\![nb_2 \sim kb_3]\!] \leq [\![mb_1 \sim kb_3]\!];$$
$$[\![mb_3 \sim nb_2]\!] \cdot [\![nb_2 \sim kb_1]\!] \leq [\![mb_3 \sim kb_1]\!];$$
$$[\![mb_3 \preccurlyeq nb_2]\!] \cdot [\![nb_2 \preccurlyeq kb_1]\!] \leq [\![mb_3 \preccurlyeq kb_1]\!].$$

We confirm that all three inequalities hold in our structures by noting that each of them is equivalent to

$$(1-w')(1-w') \leq 1-w.$$

We replace w' and w by our definitions of them to get

$$\left(1-\frac{p-3}{p-1}\right)\left(1-\frac{p-3}{p-1}\right) \leq 1-\frac{p-2}{p-1},$$

which simplifies to $5 \leq p$, which holds in our structures. That completes the proof.

Remark. How much lower than $[\![b_1 \ll b_3]\!]$ can $[\![b_1 \ll b_2]\!]$ and $[\![b_2 \ll b_3]\!]$ be? Because of F, the key constraint is that for any $m, n, k \in \mathbb{Z}^+$ and $b, b', b'' \in B$, we need $(1-[\![nb' \prec kb'']\!])(1-[\![mb \prec nb']\!]) \leq 1-[\![mb \prec kb'']\!]$ to satisfy T_5 for $\preccurlyeq$. For example, in our structure in M_{101} in which $[\![b_1 \ll b_3]\!] = w = 0.99$, we need $w' \geq 0.9$, and hence $[\![b_1 \ll b_2]\!] \geq 0.9$ and $[\![b_2 \ll b_3]\!] \geq 0.9$. In this example, it might be a problem for inferiority that there is an object such that it is true to at least the perhaps counterintuitively high degree 0.9 that it is inferior to its successor. But the structures in this appendix are simple and they have an unrealistically short sequence containing only the three bads b_1, b_2 and b_3, so this might not be a problem with longer sequences and more complex structures. As I essentially mentioned in the beginning of §8, I leave the following related, interesting question for future research: given different values of $[\![b_1 \ll b_n]\!]$ (e.g., 0.95), how low can the maximum value among all $[\![b_i \ll b_{i+1}]\!]$ for $i \in \{1, \ldots, n-1\}$ (and all $[\![b_i \ll b_{i-1}]\!]$ for $i \in \{2, \ldots, n\}$) in a finite sequence be, if the sequence might be long (e.g., $b_1, \ldots, b_{20}$), the value relations have intuitive properties (e.g., $[\![mb_4 \prec nb_{12}]\!]$ varies intuitively as m and n vary; see appendix H), and we grant T_5^r (or T_7 or T_7^r) for the relations $\preccurlyeq$, $\prec$ and $\sim$ as well as the other premises that I have granted such as F and reflexivity of $\sim$? The greater this maximum truth value must be, the stronger the sequence argument is.

G Proof of Theorem 6

Let the new structures have the same domains and function as in appendix C, and let R represent $\prec$ and $\preccurlyeq$. For all $m, n \in \mathbb{Z}^+$ and $b, b' \in B$ let

$$[\![mb_1 R nb_3]\!] = 1;$$
$$[\![nb_3 R mb_1]\!] = 0;$$
$$[\![mb_1 R nb_2]\!] = [\![mb_2 R nb_3]\!] = w;$$
$$[\![nb_2 R mb_1]\!] = [\![nb_3 R mb_2]\!] = 1 - w;$$
$$[\![mb \prec nb]\!] = \begin{cases} 1 & \text{if } m > n, \\ 0 & \text{if } m \leq n; \end{cases}$$
$$[\![mb \preccurlyeq nb]\!] = \begin{cases} 1 & \text{if } m \geq n, \\ 0 & \text{if } m < n; \end{cases}$$
$$[\![mb \sim nb']\!] = \begin{cases} 1 & \text{if } m = n \text{ and } b = b', \\ 0 & \text{otherwise;} \end{cases}$$

where $w \in (0.5, 1)$.

The only non-trivial task is to confirm that the transitivity conditions hold, so I omit the rest of the proof.

T_7 and T_8 for $\sim$ hold because of the following: To violate T_7 or T_8 for $\sim$, we need the form $\langle mb \sim nb', nb' \sim kb'', mb \sim kb'' \rangle$, where $m, n, k \in \mathbb{Z}^+$; $b, b', b'' \in B$; $[\![mb \sim nb']\!] > 0$; and $[\![nb' \sim kb'']\!] > 0$. We only get this when $m = n$, $b = b'$, $n = k$ and $b' = b''$. But then T_7 and T_8 for $\sim$ hold because $\min(1, 1) \leq 1$ and $1 + 1 - 1 \leq 1$.

T_7 and T_8 for R hold when the form $\_bR\_b$ is on the left-hand side of the inequality in the transitivity condition because to then get the form $\langle aRb, bRc, aRc \rangle$, we need (i) $\langle \_bR\_b, \_bR\_b', \_bR\_b' \rangle$ or (ii) $\langle \_b'R\_b, \_bR\_b, \_b'R\_b \rangle$. In either case, T_7 and T_8 for R hold because of the following: If $b \neq b'$, then the truth value that R maps to is independent of m and n, and for any $w_1, w_2 \in [0, 1]$, $\min(w_1, w_2) \leq w_2$ and $w_1 + w_2 - 1 \leq w_2$. If $b = b'$, there is no difference between (i) and (ii); we get $\langle mbRnb, nbRkb, mbRkb \rangle$. To violate T_7 or T_8 for R, we need $[\![mbRnb]\!] > 0$, $[\![nbRkb]\!] > 0$, and $[\![mbRkb]\!] < 1$. So, to violate T_7 or T_8 for $\prec$, we need $m > n$, $n > k$ and $m \leq k$, which is a contradiction. To violate T_7 or T_8 for $\preccurlyeq$, we need $m \geq n$, $n \geq k$ and $m < k$, which is also a contradiction.

It remains to confirm T_7 and T_8 for R when nothing on the left-hand side of T_7 or T_8 has the form $\_bR\_b$. In this case, to violate T_7, both arguments of the min function in T_7 need to be at least w for the antecedent ($bRa \leq aRb$ and $cRb \leq bRc$) of T_7 to hold. The only combination of arguments which are at least w with the form $\langle aRb, bRc \rangle$ is $\langle mb_1 Rnb_2, nb_2 Rkb_3 \rangle$. But then we get $\min(\llbracket mb_1Rnb_2\rrbracket, \llbracket nb_2Rkb_3\rrbracket) \leq \llbracket mb_1Rkb_3\rrbracket = 1$, which holds, so T_7 for R is confirmed.

To violate T_8, the left-hand side of the inequality in T_8 must be greater than 0. There are three such cases in which the terms on the left-hand side have the form $\langle aRb, bRc\rangle$. In these cases, T_8 implies the following for any $m, n, k \in \mathbb{Z}^+$:

$$\begin{aligned}
&\llbracket mb_1Rkb_3\rrbracket + \llbracket kb_3Rnb_2\rrbracket - 1 \leq \llbracket mb_1Rnb_2\rrbracket, \text{ i.e., } 1 + 1 - w - 1 \leq w;\\
&\llbracket nb_2Rmb_1\rrbracket + \llbracket mb_1Rkb_3\rrbracket - 1 \leq \llbracket nb_2Rkb_3\rrbracket, \text{ i.e., } 1 - w + 1 - 1 \leq w;\\
&\llbracket mb_1Rnb_2\rrbracket + \llbracket nb_2Rkb_3\rrbracket - 1 \leq \llbracket mb_1Rkb_3\rrbracket, \text{ i.e., } w + w - 1 \leq 1.
\end{aligned}$$

These inequalities hold so T_8 for R is confirmed. That concludes the proof.

Remark. T_8 holds in a structure that is exactly like those described so far in this appendix except that $w = \frac{1}{2}$. In that case, we would only use the three truth values in $\mathcal{W}_3 = \left\{0, \frac{1}{2}, 1\right\}$, so we would get a result like Theorem 6 in M_3 about only T_8.

H A more intuitive structure

I assume M_∞ and present a structure $\mathcal{S}$ in which it is true to degree 0.7 that the first object is inferior to the last object, but in which there is no object such that it is true to at least degree 0.7 that it is inferior to its successor. It is easy to confirm that $\sim$ and $\preccurlyeq$ are reflexive, $\prec$ is irreflexive, and F and $C_\preccurlyeq$ hold in $\mathcal{S}$, so I omit those exercises. I confirm the inferiority relationships and present a partial demonstration of that T_8 holds for $\preccurlyeq$.

$\mathcal{S}$ has the same domains and function as in appendix C. Let R represent $\prec$ and $\preccurlyeq$. For all $m, n \in \mathbb{Z}^+$, let

$$\begin{aligned}
&\llbracket nb_2Rmb_1\rrbracket = \llbracket nb_3Rmb_2\rrbracket = \begin{cases} 0 & \text{if } m \geq n,\\ 0.5\left(1 + \frac{1}{m+1}\right)\frac{\sqrt{n-m}}{\sqrt{n}} & \text{if } m < n;\end{cases}\\
&\llbracket mb_1Rnb_2\rrbracket = 1 - \llbracket nb_2Rmb_1\rrbracket;\\
&\llbracket mb_2Rnb_3\rrbracket = 1 - \llbracket nb_3Rmb_2\rrbracket;\\
&\llbracket nb_3Rmb_1\rrbracket = \begin{cases} 0 & \text{if } m \geq n,\\ 0.3\left(1 + \frac{1}{m+1}\right)\frac{\sqrt{n-m}}{\sqrt{n}} & \text{if } m < n;\end{cases}\\
&\llbracket mb_1Rnb_3\rrbracket = 1 - \llbracket nb_3Rmb_1\rrbracket;
\end{aligned}$$

and for $b, b' \in B$, define $\llbracket mb \prec nb\rrbracket$, $\llbracket mb \preccurlyeq nb\rrbracket$ and $\llbracket mb \sim nb'\rrbracket$ as in appendix G.

The following are explanatory comments on the two most important parts of the structure, namely $\left(1 + \frac{1}{m+1}\right)$ and $\frac{\sqrt{n-m}}{\sqrt{n}}$: Without $\left(1 + \frac{1}{m+1}\right)$ it would be equally true that 1 b_1-object is worse than any number of b_3-objects as that 1 billion b_1-objects are worse than any number of b_3-objects, which one might find counterintuitive. The part $\left(1+\frac{1}{m+1}\right)$ ensures that as the number m increases, it becomes truer that m b_1-objects are worse than any number of b_3-objects, which seems intuitive. It also ensures that as m approaches infinity, the truth value of that m b_1-objects are worse than any number of b_3-objects approaches a limit (the limit is set by the number 0.3; the limit becomes $1-0.3$). The part $\frac{\sqrt{n-m}}{\sqrt{n}}$ makes it so that for any given m, $[\![mb_1 \prec nb_3]\!]$ decreases and approaches a limit as n increases. Whether to use $\frac{\sqrt{n-m}}{\sqrt{n}}$ or the simpler $\frac{n-m}{n}$ seems to be inessential and simply a matter of what looks intuitive. All of this also applies to comparisons of b_1 to b_2 and of b_2 to b_3, except that 0.5 is used instead of 0.3.

It is true to degree 0.7 that b_1 is inferior to b_3, but there is no object such that it is true to at least degree 0.7 that it is inferior to an adjacent object in the sequence:

$$
\begin{aligned}
&[\![b_1 \ll b_3]\!] = \lim_{m\to\infty}\lim_{n\to\infty}\left(1-0.3\left(1+\frac{1}{m+1}\right)\frac{\sqrt{n-m}}{\sqrt{n}}\right) = 0.7;\\
&[\![b_1 \ll b_2]\!] = [\![b_2 \ll b_3]\!] =\\
&\qquad \lim_{m\to\infty}\lim_{n\to\infty}\left(1-0.5\left(1+\frac{1}{m+1}\right)\frac{\sqrt{n-m}}{\sqrt{n}}\right) = 0.5;\\
&[\![b_3 \ll b_1]\!] = [\![b_3 \ll b_2]\!] = [\![b_2 \ll b_1]\!] = 0.
\end{aligned}
$$

To confirm T_8 for $\preccurlyeq$, we need to confirm that for all $m, n, k \in \mathbb{Z}^+$ and $b, b', b'' \in B$,

$$[\![mb \preccurlyeq nb']\!] + [\![nb' \preccurlyeq kb'']\!] - 1 \leq [\![mb \preccurlyeq kb'']\!]. \tag{H1}$$

There are many cases such as when $b = b' \neq b''$ and $m = k < n$. I find that H1 holds in all cases so that T_8 of $\preccurlyeq$ holds in $\mathcal{S}$. But it is a lengthy exercise to go through all cases so I only confirm the six most difficult cases here:

Case 1 $[\![mb_1 \preccurlyeq nb_2]\!] + [\![nb_2 \preccurlyeq kb_3]\!] - 1 \leq [\![mb_1 \preccurlyeq kb_3]\!]$, when $m < n < k$;

Case 2 $[\![mb_3 \preccurlyeq nb_1]\!] + [\![nb_1 \preccurlyeq kb_2]\!] - 1 \leq [\![mb_3 \preccurlyeq kb_2]\!]$, when $n < k < m$;

Case 3 $[\![mb_2 \preccurlyeq nb_3]\!] + [\![nb_3 \preccurlyeq kb_1]\!] - 1 \leq [\![mb_2 \preccurlyeq kb_1]\!]$, when $k < m < n$;

Case 4 $[\![mb_1 \preccurlyeq nb_3]\!] + [\![nb_3 \preccurlyeq kb_2]\!] - 1 \leq [\![mb_1 \preccurlyeq kb_2]\!]$, when $m < k < n$;

Case 5 $[\![mb_2 \preccurlyeq nb_1]\!] + [\![nb_1 \preccurlyeq kb_3]\!] - 1 \leq [\![mb_2 \preccurlyeq kb_3]\!]$, when $n < m < k$;

Case 6 $[\![mb_3 \preccurlyeq nb_2]\!] + [\![nb_2 \preccurlyeq kb_1]\!] - 1 \leq [\![mb_3 \preccurlyeq kb_1]\!]$, when $k < n < m$.

We can deal with cases 1, 2 and 3 at the same time because they are equivalent. For example, case 1 becomes

$$
\begin{aligned}
0 \leq 0.5&\left(1+\frac{1}{m+1}\right)\frac{\sqrt{n-m}}{\sqrt{n}}\\
&+0.5\left(1+\frac{1}{n+1}\right)\frac{\sqrt{k-n}}{\sqrt{k}} - 0.3\left(1+\frac{1}{m+1}\right)\frac{\sqrt{k-m}}{\sqrt{k}},
\end{aligned} \tag{H2}
$$

where $m < n < k$. And case 2 becomes

$$0 \leq 0.5\left(1+\frac{1}{k+1}\right)\frac{\sqrt{m-k}}{\sqrt{m}} - 0.3\left(1+\frac{1}{n+1}\right)\frac{\sqrt{m-n}}{\sqrt{m}} + 0.5\left(1+\frac{1}{n+1}\right)\frac{\sqrt{k-n}}{\sqrt{k}}, \quad \text{(H3)}$$

where $n < k < m$. To notice that the two cases are equivalent, in H2 and its $m < n < k$, rename m to n, n to k, and k to m to get H3 and its $n < k < m$. The way to get from case 2 to case 3 and from case 3 to case 1 is analogous. So we can confirm T_8 of $\preccurlyeq$ for cases 1, 2 and 3 by confirming it for case 2, which I will do by checking that H3 holds for all $m, n, k \in \mathbb{Z}^+$ such that $n < k < m$.[33]

To minimise the right-hand side of H3 for any constant $k \geq 2$, m should be as small as possible and n should be as large as possible; that is, $m = k+1$ and $n = k-1$. The reason is that when $1 \leq n < k < m$, and n, k and m are real numbers, the first-order partial derivatives of the right-hand side of H3 with respect to m and n are positive and negative, respectively.

The partial derivative of the right-hand side of H3 with respect to m is

$$\frac{0.25k(k+2)}{m\sqrt{m}(k+1)\sqrt{m-k}} - \frac{0.15n(n+2)}{m\sqrt{m}(n+1)\sqrt{m-n}}. \quad \text{(H4)}$$

To check that H4 is positive when $1 \leq n < k < m$, confirm the following inequality for such n, k and m:

$$\frac{0.25k(k+2)}{m\sqrt{m}(k+1)\sqrt{m-k}} > \frac{0.15n(n+2)}{m\sqrt{m}(n+1)\sqrt{m-n}}. \quad \text{(H5)}$$

On both sides of H5, multiply by $m\sqrt{m}$, $k+1$ and $n+1$, and divide by 0.15 to get

$$\frac{\frac{5}{3}k(k+2)(n+1)}{\sqrt{m-k}} > \frac{n(n+2)(k+1)}{\sqrt{m-n}}. \quad \text{(H6)}$$

When $1 \leq n < k < m$, the following holds: The denominator on the left-hand side of H6 is less than the denominator on the right-hand side, and after expanding the products in the numerators, one can see that the numerator on the left-hand side is greater than the numerator on the right-hand side. H5 is confirmed, so the partial derivative of the right-hand side of H3 with respect to m is positive when $1 \leq n < k < m$.

The partial derivative of the right-hand side of H3 with respect to n is

$$\frac{0.15\left(n^2+n+2\right)+0.3m}{\sqrt{m}\sqrt{m-n}(1+n)^2} - \frac{0.25\left(n^2+n+2\right)+0.5k}{\sqrt{k}\sqrt{k-n}(1+n)^2}. \quad \text{(H7)}$$

[33] Thanks to Magnus Vinding for explaining how one can check H3 and several of the other inequalities below.

To confirm that H7 is less than 0 when $1 \leq n < k < m$, note that for such n, k and m,

$$\frac{0.15\left(n^2+n+2\right)}{\sqrt{m}\sqrt{m-n}(1+n)^2} < \frac{0.25\left(n^2+n+2\right)}{\sqrt{k}\sqrt{k-n}(1+n)^2} \tag{H8}$$

because $0.15 < 0.25$ and $k < m$, and then confirm

$$\frac{0.3m}{\sqrt{m}\sqrt{m-n}(1+n)^2} < \frac{0.5k}{\sqrt{k}\sqrt{k-n}(1+n)^2}, \tag{H9}$$

which we can do by simplifying and rearranging H9 to

$$\begin{aligned}
\frac{m}{\sqrt{m}\sqrt{m-n}} &< \frac{5}{3}\frac{k}{\sqrt{k}\sqrt{k-n}};\\
m\left(k-n\right) &< \left(\frac{5}{3}\right)^2 k\left(m-n\right);\\
\frac{16}{9}kn + kn &< \frac{16}{9}km + mn.
\end{aligned}$$

This holds when $1 \leq n < k < m$ because $\frac{16}{9}kn < \frac{16}{9}km$ and $kn < mn$. So the partial derivative of the right-hand side of H3 with respect to n is negative when $1 \leq n < k < m$.

In H3, replace m by $k+1$ and n by $k-1$ to get

$$0 \leq 0.5\left(1+\frac{1}{k+1}\right)\frac{1}{\sqrt{k+1}} - 0.3\left(1+\frac{1}{k}\right)\frac{\sqrt{2}}{\sqrt{k+1}} + 0.5\left(1+\frac{1}{k}\right)\frac{1}{\sqrt{k}}, \tag{H10}$$

where $k \geq 2$. H10 holds for all $k \geq 2$ because $\frac{0.3\sqrt{2}}{\sqrt{k+1}} < \frac{0.5}{\sqrt{k}}$ for all $k \geq 2$. Thus, H3 holds for all $m, n, k \in \mathbb{Z}^+$ such that $n < k < m$. T_8 of $\preccurlyeq$ for cases 1, 2 and 3 is confirmed.

Cases 4, 5 and 6 can be treated all at once for the same reason as cases 1, 2 and 3. I will confirm T_8 of $\preccurlyeq$ for cases 4, 5 and 6 by confirming the following inequality based on case 5:

$$0 \leq 1 - 0.5\left(1+\frac{1}{m+1}\right)\frac{\sqrt{k-m}}{\sqrt{k}} - 0.5\left(1+\frac{1}{n+1}\right)\frac{\sqrt{m-n}}{\sqrt{m}} + 0.3\left(1+\frac{1}{n+1}\right)\frac{\sqrt{k-n}}{\sqrt{k}}, \tag{H11}$$

where $n, k, m \in \mathbb{Z}^+$ and $n < m < k$. The partial derivative of the right-hand side of H11 with respect to k is negative when n, m and k are real numbers, and $1 \leq n < m < k$. So k should be as large as possible to minimise the right-hand side of H11. The limit of the right-hand side of H11 as k goes to infinity is the right-hand side of

$$0 \leq 1 - 0.5\left(1 + \frac{1}{m+1}\right) - 0.5\left(1 + \frac{1}{n+1}\right)\frac{\sqrt{m-n}}{\sqrt{m}} + 0.3\left(1 + \frac{1}{n+1}\right). \quad \text{(H12)}$$

H12 holds for all $n, m \in \mathbb{Z}^+$ such that $n < m$, as one can see by expanding the brackets in H12 and simplifying. So T_8 of $\preccurlyeq$ is confirmed for case 5 and thus also for cases 4 and 6.

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